HYPERTORIC GEOMETRY AND GROMOV-WITTEN THEORY

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Abstract. We study Gromov-Witten theory of hypertoric Deligne-Mumford stacks from two points of view. From the viewpoint of representation theory, we calculate the operator of small quantum product by a divisor, following [7], [39], [40]. From the viewpoint of Lawrence toric geometry, we compare Gromov-Witten invariants of a hypertoric Deligne-Mumford stack with those of its associated Lawrence toric stack.

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1. Introduction

1.1. **Background.** In recent years, Gromov-Witten theory of symplectic varieties has been proved to have deep connections to geometric representation theory by the work of Braverman-Maulik-Okounkov [7], Maulik-Okounkov [39]. In Braverman-Maulik-Okounkov [7], the authors prove a quantum product formula by a divisor class for a smooth symplectic variety X, which is close related to the symplectic resolution of singularities.

Symplectic Deligne-Mumford stacks are the corresponding symplectic resolutions in the stacky world. The Gromov-Witten theory of smooth Deligne-Mumford stacks has been developed in algebraic category by Abramovich-Graber-Vistoli [1] and in symplectic category by Chen-Ruan [11]. It is interesting to study the quantum cohomology of symplectic Deligne-Mumford stacks and explore the relationship to representation theory. Hypertoric Deligne-Mumford stacks (hypertoric DM stack), defined in [28] using stacky hyperplane arrangements, are hyperkähler analogue of Kähler toric Deligne-Mumford stacks. They provide important examples of smooth symplectic Deligne-Mumford stacks. In this paper we study the quantum cohomology of hypertoric Deligne-Mumford stacks, and generalize the quantum product formula by a divisor class in [7] to hypertoric cases. Note that for the smooth hypertoric varieties, the quantum cohomology has been studied in [40].

1.2. **Hypertoric Deligne-Mumford stacks.** Let N be a finitely generated abelian group of rank d and $N \to \overline{N}$ the natural projection modulo torsion. Let $\beta : \mathbb{Z}^m \to N$ be a homomorphism determined by a collection of nontorsion integral vectors $\{b_1, \cdots, b_m\} \subseteq N$. The map β is required to have finite cokernel. The Gale dual (in the sense of [6]) of β is denoted by $\beta^\vee : (\mathbb{Z}^m)^* \to DG(\beta)$. A *generic* element θ in $DG(\beta)$ and the vectors $\{\overline{b}_1, \cdots, \overline{b}_m\}$ determine a hyperplane arrangement $\mathcal{H} = (H_1, \cdots, H_m)$ in $N^*_{\mathbb{R}}$. We call $\mathcal{A} := (N, \beta, \theta)$ a *stacky hyperplane arrangement*.

There is a stacky fan $\mathbf{\Sigma}_\theta$ associated to the stacky hyperplane arrangement $\mathcal{A}$, and the associated toric Deligne-Mumford stack $\chi(\mathbf{\Sigma}_\theta)$ is called the *Lawrence toric Deligne-Mumford stack*. The stack $\chi(\mathbf{\Sigma}_\theta) = [U/G]$ is a quotient stack, where $U \subset \mathbb{C}^{2m} = T^*\mathbb{C}^m$ is an open subvariety determined by the irrelevant ideal of the Lawrence fan Σ_θ, G is a finitely generated abelian group, and the G-action on U is determined by the stacky fan $\mathbf{\Sigma}_\theta$. The hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is defined as the quotient stack $[Y/G]$, where $Y \subset U$ is a closed subvariety determined by a prime ideal in $T^*\mathbb{C}^m$,

see §2.1. Hence the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is a closed substack of the Lawrence toric Deligne-Mumford stack $\chi(\mathbf{\Sigma}_\theta)$. The topology of $X_{\mathcal{A}}$ is determined by the hyperplane arrangement $\mathcal{A}$. In $\mathcal{H}$, the hyperplanes H_i bound finite number of polytopes whose corresponding toric Deligne-Mumford stacks in the sense of [6], [27], form the core $C(X_{\mathcal{A}})$ of $X_{\mathcal{A}}$. The core $C(X_{\mathcal{A}})$ is the deformation retract of $X_{\mathcal{A}}$.

The map β determines a multi-fan Δ_β in $N_{\mathbb{R}}$, which consists of cones generated by linearly independent subsets $\{\overline{b}_{i_1}, \cdots, \overline{b}_{i_k}\}$ in $\overline{N}$ for $\{i_1, \cdots, i_k\} \subset \{1, \cdots, m\}$, see §2.1.5. We define a set $\mathrm{Box}(\Delta_\beta)$ consisting of all pairs (v, σ), where σ is a cone in the multi-fan Δ_β, $v \in N$ such that $\overline{v} = \sum_{\rho_i \subset \sigma} \alpha_i \overline{b}_i$ for $0 < \alpha_i < 1$. For $(v, \sigma) \in \mathrm{Box}(\Delta_\beta)$ we consider a closed substack of $X_{\mathcal{A}}$ given by the quotient stacky hyperplane arrangement $\mathcal{A}(\sigma)$. The inertia stack of $X_{\mathcal{A}}$ is the disjoint union of all such closed substacks, see §2.1.5. The Chen-Ruan cohomology $H^*_{\mathrm{CR}}(X_{\mathcal{A}}) = H^*(IX_{\mathcal{A}})$ is the same as the cohomology of inertia stack up to the Chen-Ruan degree shifting. The stack $X_{\mathcal{A}}$ admits a torus $\mathbb{T} := (\mathbb{C}^\times)^m \times \mathbb{C}^\times$ action, where the $(\mathbb{C}^\times)^m$-action is induced from the standard action on $T^*\mathbb{C}^m$ and the extra $\mathbb{C}^\times$ acts by scaling the fibre. We consider the $\mathbb{T}$-equivariant Chen-Ruan cohomology $H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}})$ of $X_{\mathcal{A}}$. Similar to the main result in [28], the $\mathbb{T}$-equivariant Chen-Ruan cohomology $H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}})$ is described by the matroid complex of the multi fan Δ_β, see §2.2.

1.3. **Symplectic resolution and the Steinberg correspondence.** The genericness of $\theta \in DG(\beta)$ implies that the hyperplane arrangement $\mathcal{H}$ is simple. Hence the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is smooth. There is a nowhere vanishing symplectic form ω on $X_{\mathcal{A}}$ induced from the standard symplectic form on $T^*\mathbb{C}^m$, thus $X_{\mathcal{A}}$ is a smooth symplectic Deligne-Mumford stack.

If we choose $\theta = 0$, then the hyperplanes H_i in $\mathcal{H}$ all pass through the origin. We denote the corresponding hypertoric stack by X^0. It is not Deligne-Mumford, and is a singular stack. Let $\overline{X}^0$ be its good moduli space in the sense of Alper [2]. There is a contraction map

$$\phi : X_{\mathcal{A}} \to \overline{X}^0$$

which contracts the core $C(X_{\mathcal{A}})$ to singular points. Hence the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is a symplectic resolution of the singular symplectic variety $\overline{X}^0$.

Let $X_{\mathcal{A}} \times_{\overline{X}^0} X_{\mathcal{A}}$ be the fibre product. The components whose dimension are the same as $X_{\mathcal{A}}$ are called the *Steinberg stack*. Let $\mathbf{Z}$ be the union of all such components and $I\mathbf{Z}$ the inertia stack of $\mathbf{Z}$. Then $I\mathbf{Z}$ gives a correspondence on the $\mathbb{T}$-equivariant Chen-Ruan cohomology of $X_{\mathcal{A}}$ as follows. First the fibre product $X_{\mathcal{A}} \times_{\overline{X}^0} X_{\mathcal{A}} \subset X_{\mathcal{A}} \times X_{\mathcal{A}}$ is inside the product of $X_{\mathcal{A}}$. By Poincaré duality, the inertia stack $[I\mathbf{Z}]$, taken as a cycle in $H_*(I(X_{\mathcal{A}} \times X_{\mathcal{A}}))$, yields a class in the cohomology $H^*_{\mathbb{T}}(I(X_{\mathcal{A}} \times X_{\mathcal{A}}))$. Let

$$\mathrm{inv} : IX_{\mathcal{A}} \to IX_{\mathcal{A}}$$

be the involution sending $(x, g) \mapsto (x, g^{-1})$ for $x \in X_{\mathcal{A}}$, $g \in \mathrm{Aut}(x)$. The Steinberg correspondence

$$I\mathbf{Z} : H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) \to H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) \tag{1.1}$$

is given by:

$$I\mathbf{Z}(\alpha) = \mathrm{inv}^\star Ip_{2,\star}(Ip_1^\star \alpha \cup [I\mathbf{Z}]),$$

where $p_i : X_{\mathcal{A}} \times X_{\mathcal{A}} \to X_{\mathcal{A}}$ are the projections for $i = 1, 2$ and $Ip_i : IX_{\mathcal{A}} \times IX_{\mathcal{A}} \to IX_{\mathcal{A}}$ are the projections on the corresponding inertia stacks. The Steinberg correspondence

gives rise to an endomorphism on the $\mathbb{T}$-equivariant Chen-Ruan cohomology of $X_{\mathcal{A}}$. We prove that the equivariant quantum product by divisor classes of $X_{\mathcal{A}}$ is given by Steinberg correspondence in (1.1).

1.4. **Small quantum product.** Let $\mathrm{NE}(X_{\mathcal{A}}) \subset H_2(X_{\mathcal{A}}, \mathbb{R})$ be the cone generated by classes of effective curves, and $\mathrm{NE}(X_{\mathcal{A}})_{\mathbb{Z}} = \{d \in H_2(X_{\mathcal{A}}, \mathbb{Z}) : d \in \mathrm{NE}(X_{\mathcal{A}})\}$. We denote $R_{\mathbb{T}} := H^*_{\mathbb{T}}(pt)$ which is the $\mathbb{T}$-equivariant cohomology of a point, and let

$$R_{\mathbb{T}}[\![Q]\!] = \left\{ \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} a_d Q^d : a_d \in R \right\}.$$

Here Q is a so-called *Novikov variable* [37, III 5.2.1]. For $\gamma_i, \gamma_j \in H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}})$, the small equivariant quantum product is defined by:

$$(\gamma_i \star \gamma_j, \gamma_k) = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} Q^d \langle \gamma_i, \gamma_j, \gamma_k \rangle^{X_{\mathcal{A}}}_{0,3,d}. \tag{1.2}$$

Here $\langle -, -, -, \rangle^{X_{\mathcal{A}}}_{0,3,d}$ are 3-point genus 0 $\mathbb{T}$-equivariant Gromov-Witten invariants of $X_{\mathcal{A}}$. Genus 0 $\mathbb{T}$-equivariant Gromov-Witten invariants of $X_{\mathcal{A}}$ are defined as integrations against the $\mathbb{T}$-equivariant virtual fundamental class $[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)]^{\mathrm{virt}}$ of the moduli stack $\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)$ of twisted stable maps from genus zero twisted curve with n-marked points to $X_{\mathcal{A}}$ of degree $d \in H_2(X_{\mathcal{A}}, \mathbb{Z})$. The existence of a holomorphic symplectic form on $X_{\mathcal{A}}$ implies that the obstruction sheaf of $\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)$ has a trivial quotient, i.e. a cosection in the sense of [32]. This yields a *reduced virtual fundamental class* $[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)]^{\mathrm{red}}$ for $\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)$. This reduced class satisfies

$$[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)]^{\mathrm{virt}} = \hbar \cdot [\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)]^{\mathrm{red}}.$$

Here $\hbar$ is the equivariant parameter for the $\mathbb{C}^\times \subset \mathbb{T} = T \times \mathbb{C}^\times$ action. See §2.3.2 for more details.

Let u_i be a divisor class for $X_{\mathcal{A}}$. The divisor equation of Gromov-Witten invariants reduces the determination of quantum product by u_i to two-point invariants of $X_{\mathcal{A}}$. By the relation between virtual fundamental cycle and reduced virtual cycle, we are reduced to calculate the pushforward by the evaluation map $\mathbf{ev} : \overline{\mathcal{M}}_{0,2}(X_{\mathcal{A}}, d) \to IX_{\mathcal{A}} \times IX_{\mathcal{A}}$:

$$\Gamma_2 := \mathbf{ev}_\star([\overline{\mathcal{M}}_{0,2}(X_{\mathcal{A}}, d)]^{\mathrm{red}}) \subset H_*(IX_{\mathcal{A}} \times IX_{\mathcal{A}}).$$

Components of Γ_2 all have Chen-Ruan degree $\dim(X_{\mathcal{A}})$ in $H^*_{\mathrm{CR}}(X_{\mathcal{A}} \times X_{\mathcal{A}})$, which we call *Lagrangian cycle* in orbifold sense.

Following [39], such cycle Γ_2 supports on $I\mathbf{Z} \subset I(X_{\mathcal{A}} \times X_{\mathcal{A}})$, and is called the *Steinberg Variety*. The correspondence

$$\Gamma_2 : H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) \to H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) \tag{1.3}$$

is given by:

$$\Gamma_2(\gamma) = \mathrm{inv}^\star Ip_{2,\star}(Ip_1^\star \gamma \cup \Gamma_2),$$

where $Ip_i : IX_{\mathcal{A}} \times IX_{\mathcal{A}} \to IX_{\mathcal{A}}$ is the projection for $i = 1, 2$, and Γ_2 is taken as a cohomology class in $H^*_{\mathbb{T}}(IX_{\mathcal{A}} \times IX_{\mathcal{A}})$. This is the Steinberg correspondence in (1.1). Recall that the index set of the components of $IX_{\mathcal{A}}$ is given by I. We can write

$$\Gamma_2 := \bigoplus_{\substack{(f_1,f_2):\\ f_1,f_2 \in I}} \Gamma_{f_1,f_2}.$$

A circuit $S \subset \mathcal{A}$ is a minimal subset $S \subset \{1, \cdots, m\}$ such that $\{\overline{b}_i | i \in S\}$ are linearly dependent. For any circuit $S \subset \mathcal{A}$, there is an associated curve class $\beta^S \in H_2(X_{\mathcal{A}})$ defined as follows: there is a splitting $S = S^+ \cup S^-$ and positive integers w_i such that

$$\beta^S := \sum_{i \in S^+} w_i e_i - \sum_{i \in S^-} w_j e_j.$$

Set $\mathbf{w} := \{w_i | i \in S\}$.

The deformation of the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is classified by the image of the symplectic form in the second cohomology $H^2(X_{\mathcal{A}}, \mathbb{C})$, which is isomorphic to $DG(\beta)_{\mathbb{C}}$. In §3 we construct the following diagram:

$$\begin{tikzcd} X_{\mathcal{A}} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\phi"'] \\ 0 \arrow[r] & DG(\beta)_{\mathbb{C}} \end{tikzcd} \tag{1.4}$$

such that for *sub-regular parameters* $\lambda \in DG(\beta)_{\mathbb{C}}$ the deformation $X_\lambda := \mathcal{M}_\lambda(\mathcal{H})$ has nice properties. The stack X_λ contains a closed substack $\overline{\mathcal{M}}^S$ for each circuit $S \subset \mathcal{A}$, which is a weighted projective bundle over an affine base and the fibre normal bundle is the cotangent bundle of the weighted projective stack $\mathbb{P}_{\mathbf{w}}^{|S|-1}$. All curve classes of X_λ lie in $\overline{\mathcal{M}}^S$, see §3. Deformation invariance of Gromov-Witten invariants implies that Gromov-Witten invariants in class β^S can be computed by Gromov-Witten invariants of $T^*\mathbb{P}_{\mathbf{w}}^{|S|-1}$.

Thus components of IX_λ are all classified by the set $F = \{\frac{a_i}{w_i} | 0 \leq a_i \leq w_i\}$. Let $l(\mathbf{w}) := \mathrm{lcm}(w_i | i \in S)$. For any $f_1 = \frac{a_1}{w_i}$ and $f_2 = \frac{a_2}{w_j}$, let $\gamma(f_1, f_2) \in \{0, 1, \cdots, l(\mathbf{w}) - 1\}$ such that [1] $\langle \frac{\gamma(f_1,f_2)}{w_i} \rangle = f_1$, $\langle \frac{\gamma(f_1,f_2)}{w_j} \rangle = f_2$, and $w_k | \gamma(f_1, f_2)$ for $w_k \neq w_i, w_j$.

Theorem 1.1. *Let u_i be a divisor class of $X_{\mathcal{A}}$. Then the small equivariant quantum product formula by the divisor u_i is given by:*

$$u_i \star - = u_i \cdot - + \sum_{\substack{S \subset \mathcal{A}: \\ circuit}} \hbar \cdot \langle u_i, \beta^S \rangle \cdot (-1)^d \cdot \sum_{(f_1,\sigma),(f_2,\tau) \in Box(\Delta_\beta)} \frac{(Q^S)^{\gamma(f_1,f_2) + l(\mathbf{w}) \cdot \delta_{\gamma(f_1,f_2),0}}}{1 - (Q^S)^{l(\mathbf{w})}} \Gamma_{f_1,f_2}(-),$$

where $\Gamma_{f_1,f_2} \in H^{2n}_{CR,\mathbb{T}}(X_{\mathcal{A}} \times X_{\mathcal{A}})$ is the Steinberg correspondence in (1.3).

Theorem 1.1 is proved by a detail analysis of two point Gromov-Witten invariants of the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$. Under evaluation maps, the image of the reduced virtual fundamental cycle $[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)]^{\mathrm{red}}$ will have orbifold diegree $\dim(X_{\mathcal{A}})$ inside $I(X_{\mathcal{A}} \times X_{\mathcal{A}})$, hence the calculation of two point Gromov-Witten invariants is reduced to Steinberg correspondence in (1.3), see §5 for more details.

Although the virtual fundamental cycle $[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}}, d)]^{\mathrm{red}}$ is a routine generalization from the case when $X_{\mathcal{A}}$ is a smooth variety, the calculation of two point Gromov-Witten invariants is given by torus localization in orbifold Gromov-Witten theory, which seems to be a new technique in this setting to calculate the small quantum product by a divisor class.

[1] The existence of $\gamma(f_1, f_2)$ imposes constraints on the possible pairs (f_1, f_2).

1.5. **Lawrence toric geometry.** We use the geometry of Lawrence toric Deligne-Mumford stacks to give a formula for the small quantum cohomology of $X_{\mathcal{A}}$.

Theorem 1.1 determines the small equivariant quantum products by any divisor class $u_i \in H^2(X_{\mathcal{A}})$. In order to have a formula for the small quantum cohomology ring of $X_{\mathcal{A}}$, we prove that the small quantum ring of $X_{\mathcal{A}}$ is isomorphic to the small quantum ring of the corresponding Lawrence toric Deligne-Mumford stack $X_\theta := \chi(\mathbf{\Sigma}_\theta)$, see Corollary 6.5. This is derived from a more general result that equates the decendant Gromov-Witten invariants of X_θ with the decendant Gromov-Witten invariants of $X_{\mathcal{A}}$, see Proposition 6.4.

Recall that in the Lawrence toric fan Σ_θ, the lattice is given by N_L, and

$$\beta_L : \mathbb{Z}^{2m} \to N_L$$

is given by integral vectors $\{b_{L,1}, \cdots, b_{L,m}, b'_{L,1}, \cdots, b'_{L,m}\} \subset N_L$, see §2.1.2. The map β_L is called the Lawrence lifting of $\beta : \mathbb{Z}^m \to N$. Let

$$R_{\mathbb{T}}[[Q]][N_{\mathbf{\Sigma}_\theta}\widehat{]}$$

be the ring generated over $R_{\mathbb{T}}[[Q]]$ by symbols $\{y^c | c \in N_L\}$, with the following multiplication: for $c_1, c_2 \in N_L$,

$$y^{c_1} \star y^{c_2} = Q^{l(c_1,c_2)} y^{c_1+c_2},$$

where $l(c_1, c_2)$ is defined as follows. Suppose that $\sigma_1, \sigma_2, \sigma$ are cones in Σ_θ such that $c_1 \in \sigma_1, c_2 \in \sigma_2, c_1+c_2 \in \sigma$. We can write $c_1 = \sum_i (c_{1i} b_{L,i} + c'_{1i} b'_{L,i})$, $c_2 = \sum_i (c_{2i} b_{L,i} + c'_{2i} b'_{L,i})$, $c_1 + c_2 = \sum_i (c_{12i} b_{L,i} + c'_{12i} b'_{L,i})$, where it is understood that $c_{1i} = 0$ if $b_{L,i} \notin \sigma_1$, $c'_{1i} = 0$ if $b'_{L,i} \notin \sigma_1$, and likewise for $c_{2i}, c'_{2i}, c_{12i}, c'_{12i}$. Put

$$l(c_1, c_2) := \sum_j ((c_{1j} + c_{2j} - c_{12j}) e_j + (c'_{1j} + c'_{2j} - c'_{12j}) e'_j) \in \mathbb{Q}^m \oplus \mathbb{Q}^m.$$

Note that $l(c_1, c_2) = 0$ if c_1, c_2 belong to the same cone.

Theorem 1.2. *Let $X_{\mathcal{A}}$ be the hypertoric Deligne-Mumford stack associated to the stacky hyperplane arrangement $\mathcal{A}$. Then the equivariant small quantum cohomology of $X_{\mathcal{A}}$ is*

$$QH^*_{\mathbb{T}}(X_{\mathcal{A}}) \cong \frac{R_{\mathbb{T}}[[Q]][N_{\mathbf{\Sigma}_\theta}\widehat{]}}{\langle y^{b_{L,i}} + y^{b'_{L,i}} - \hbar | i = 1, ..., m\rangle}.$$

Theorem 1.2 is obtained by calculating the small quantum cohomology ring of the Lawrence toric Deligne-Mumford stack X_θ. The presentation above is obtained from calculations with the (extended) I-function of X_θ. The isomorphism follows from toric mirror theorem [14] and calculations of the mirror map along H^2, see §6.2.

Remark 1.3.

(1) Theorem 1.2 specializes to the calculation in [40]. If the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is a smooth variety, which means that the hyperplane arrangement $\mathcal{H}$ is unimodular, see [23], then the corresponding Lawrence toric variety X_θ is a smooth variety. There is an one-to-one correspondence between the generators $d \in H^2(X_\theta, \mathbb{Z}) \cong DG(\beta)_{\mathbb{Z}}$ and the circuits $S \subset \mathcal{A}$. The splitting $S = S^+ \cup S^-$ is actually given by $S^+ = \{i \in S | \langle D_i, d\rangle > 0\}$ and $S^- = \{i \in S | \langle D_i, d\rangle\} < 0)$. By Poincaré duality, such a generator d determines a curve class $\beta^S \in H_2(X_\theta, \mathbb{Z})$. The quantum Stanley-Reisner ideal QSR in [40, Theorem 1.1] can be obtained directly from Theorem 1.2.

(2) The presentation in Theorem 1.2 may be rewritten in the form of generators and relations, along the line of [46, Theorem 4.9].

1.6. **Further studies.** The monodromy conjecture for symplectic resolutions was formulated by Braverman-Maulik-Okounkov in [7]. Roughly speaking a compactified Kähler moduli space $\mathcal{M}$ have large radius points $0, \infty$ such that the corresponding two symplectic Deligne-Mumford stacks X_1, X_2 are birational equivalent. The derived categories of them are expected to be equivalent:

$$D^b(X_1) \cong D^b(X_2).$$

The equivalence is given by a choice of path from 0 to ∞ and thus giving a map

$$\rho : \pi_1(\mathcal{M}) \to \mathrm{Aut}(D^b(X_i)).$$

Moreover they expect that this map is considered in the level of K-theory

$$\rho : \pi_1(\mathcal{M}) \to \mathrm{Aut}(K_0(X_i)).$$

The monodromy conjecture says that the mondromy of the quantum connection $\bigtriangledown$ for X_i is the same as the above mondromy given by the equivalence on the K-theory.

Note that this conjecture is already known in the case of crepant birational transformation of toric Deligne-Mumford stacks. In [13], the crepant transformation conjecture in Gromov-Witten theory was proved for a crepant birational transformation of toric Deligne-Mumford stacks given by single wall crossing. Let $X_+ \dashrightarrow X_-$ be a crepant birational map between two smooth toric Deligne-Mumford stacks. They are derived equivalent, which is given by Fourier-Mukai transform, see [31] and [12]. In [13], the authors prove that the equivariant Fourier-Mukai transformation on the K-theory matches the analytic continuation of the I-function, hence matches the quantum connection which is determined by the I-function. Fourier-Mukai transformation depends on a choice of path in the mirror of the toric Deligne-Mumford stacks, and applying twice of Fourier-Mukai gives the monodromy. The matching of the Fourier-Mukai transform with analytic continuation of quantum connections implies that their corresponding monodromies are the same.

In [30], we will study the case of wall crossing of hypertoric Deligne-Mumford stacks by varying the stability parameters θ. The wall crossing of hypertoric Deligne-Mumford stacks is actually given by a single wall crossing of Lawrence toric Deligne-Mumford stacks studied in [13]. Hence the wall crossing is given by the Mukai type flops of hypertoric Deligne-Mumford stacks. Several authors, see [8], [9], [31], already proved that their derived categories are equivalent, and the kernel is also given by Fourier-Mukai type transform. We expect to prove that the Fourier-Mukai transform matches the analytic continuation of quantum connections of the hypertoric Deligne-Mumford stack.

The Monodromy conjecture works for any two crepant birational transformation of symplectic Deligne-Mumford stacks. One type of such birational equivalence is the Mukai type flops, which are studied by many mathematician, see for instance [8], [9], [31]. In some nice situation, their derived categories are equivalent and Fourier-Mukai type transformation gives the equivalence. These are more general cases than hypertoric Deligne-Mumford stacks. We hope that our approach in this project may shed light on proving the conjecture in more general cases.

1.7. **Outline.** The rest of this paper is organized as follows. The notion of hypertoric Deligne-Mumford stacks and their properties are reviewed in §2.1. In §2.2 we determine the equivariant Chen-Ruan cohomology of hypertoric Deligne-Mumford stacks. Gromov-Witten theory and reduced virtual fundamental cycles are reviewed in §2.3. In §3 we discuss the deformation of hypertoric Deligne-Mumford stacks by sub-regular parameter under the moment maps. We discuss the Steinberg correspondence in §4 for symplectic resolution of hypertoric Deligne-Mumford stacks. We prove Theorem 1.1 in §5; and in §6 we study Gromov-Witten invariants of hypertoric Deligne-Mumford stacks. We give a ring structure for the small equivariant quantum cohomology of hypertoric Deligne-Mumford stacks and calculate two examples.

1.8. **Set-up.** We work over the field of complex numbers. Cohomology groups are taken with rational coefficients.

N is a finitely generated abelian group of rank d.

$N \to \overline{N} := N/N_{\mathrm{tor}}$ is the natural quotient map.

$\beta : \mathbb{Z}^m \to N$ is a group homomorphism determined by sending the standard basis of $\mathbb{Z}^m$ to a collection of nontorsion integral vectors $\{b_1, \cdots, b_m\} \subseteq N$.

$\beta^\vee : \mathbb{Z}^m \to DG(\beta)$ is the Gale dual of β as constructed in [6].

For cones σ_1, σ_2 in $\mathbb{R}^d$, we use $\sigma_1 \cup \sigma_2$ to represent the set of union of the generators of σ_1 and σ_2. For a positive integer m, we use $[m]$ to represent the set $\{1, \cdots, m\}$.

For a rational number c, let $\langle c \rangle$ denote the fractional part of c, $\lceil c \rceil$ the ceiling of c, and $\lfloor c \rfloor$ the floor of c.

For a stacky hyperplane arrangement $\mathcal{A}$, we denote by $X_{\mathcal{A}}$ the associated hypertoric Deligne-Mumford stack throughout the paper. For a sub-regular parameter λ under the moment map, we denote by $X_\lambda := \mathcal{M}_\lambda(\mathcal{H})$ the deformation of $X_{\mathcal{A}}$ by this sub-regular parameter λ. We set $\mathrm{Box} := \mathrm{Box}(\Delta_\beta)$, the box element of the multi-fan Δ_β.

For a smooth Deligne-Mumford stack X with a torus $\mathbb{T}$-action, we use $H^*_{\mathrm{CR},\mathbb{T}}(X)$ to represent the equivariant Chen-Ruan cohomology of X, $QH^*_{\mathbb{T}}(X)$ the equivariant small quantum cohomology ring, and $QH^*_{\mathbb{T},\mathrm{big}}(X)$ the equivariant big quantum cohomology ring. The Chen-Ruan orbifold cup product is denoted by $\cdot$, the small quantum product is denoted by $\star$, and the big quantum product is denoted by $\star_{\mathrm{big}}$.

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2. Preliminaries

2.1. **Hypertoric geometry.** We recall the definition of hypertoric Deligne-Mumford stacks in sense of [28].

2.1.1. *Stacky hyperplane arrangements.* We introduce stacky hyperplane arrangements and explain their relations to extended stacky fans.

Let N be a finitely generated abelian group of rank d and $\beta : \mathbb{Z}^m \to N$ a map given by nontorsion integral vectors $\{b_1, \cdots, b_m\}$. We have the following exact sequences:

$$0 \longrightarrow DG(\beta)^* \xrightarrow{(\beta^\vee)^*} \mathbb{Z}^m \xrightarrow{\beta} N \longrightarrow Coker(\beta) \longrightarrow 0, \tag{2.1}$$

$$0 \longrightarrow N^* \longrightarrow \mathbb{Z}^m \xrightarrow{\beta^\vee} DG(\beta) \longrightarrow Coker(\beta^\vee) \longrightarrow 0, \tag{2.2}$$

where $\beta^\vee$ is the Gale dual of β (see [6]). The map $\beta^\vee$ is given by the integral vectors $\{a_1, \cdots, a_m\} \subseteq DG(\beta)$. Choose a generic element $\theta \in DG(\beta)$ which lies in the image of $\beta^\vee$ and let $\psi := (r_1, \cdots, r_m)$ be a lifting of θ in $\mathbb{Z}^m$ such that $\theta = -\beta^\vee\psi$. Note that θ is generic if and only if it is not in any hyperplane of the configuration determined by $\beta^\vee$ in $DG(\beta)_{\mathbb{R}}$. Let $M = N^*$ be the dual of N. $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ is a d-dimensional $\mathbb{R}$-vector space. Associated to θ there is a hyperplane arrangement $\mathcal{H} = \{H_1, \cdots, H_m\}$ in $M_{\mathbb{R}}$ defined by H_i the hyperplane

$$H_i := \{v \in M_{\mathbb{R}} | < b_i, v > +r_i = 0\} \subset M_{\mathbb{R}}. \tag{2.3}$$

So (2.3) determines hyperplane arrangements in $M_{\mathbb{R}}$, up to translation induced by the choice of the lifting $\psi := (r_1, \cdots, r_m)$.

Definition 2.1. We call $\mathcal{A} := (N, \beta, \theta)$ a *stacky hyperplane arrangement*.

It is well-known that hyperplane arrangements determine the topology of hypertoric varieties [5]. Let

$$\mathbf{\Gamma} = \bigcap_{i=1}^{m} F_i, \text{ where } F_i = \{v \in M_{\mathbb{R}} | < b_i, v > +r_i \geq 0\}.$$

Let Σ be the normal fan of $\mathbf{\Gamma}$ in $M_{\mathbb{R}} = \mathbb{R}^d$ with one dimensional rays generated by $\overline{b}_1, \cdots, \overline{b}_n$. By reordering, we may assume that $H_1, \cdots, H_n$ are the hyperplanes that bound the polytope $\mathbf{\Gamma}$, and $H_{n+1}, \cdots, H_m$ are the other hyperplanes. Then we have an extended stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ in the sense of [27], where $\beta : \mathbb{Z}^m \to N$ is given by $\{b_1, \cdots, b_n, b_{n+1}, \cdots, b_m\} \subset N$, and $\{b_{n+1}, \cdots, b_m\}$ are the extra data.

By [27], the extended stacky fan $\mathbf{\Sigma}$ determines a toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$. Its coarse moduli space is the toric variety corresponding to the normal fan of $\mathbf{\Gamma}$. According to [5], a hyperplane arrangement $\mathcal{H}$ is *simple* if the codimension of the nonempty intersection of any l hyperplanes is l. A hypertoric variety is the coarse moduli space of an *orbifold* if the corresponding hyperplane arrangement is simple.

Remark 2.2. Consider the map $\mathbb{Z}^n \to N$ given by $\{b_1, ..., b_n\}$. Then $(N, \Sigma, \mathbb{Z}^n \to N)$ is a stacky fan in the sense of [6]. The associated toric Deligne-Mumford stack is isomorphic to $\mathcal{X}(\mathbf{\Sigma})$.

2.1.2. *Lawrence toric Deligne-Mumford stacks.* Consider the Gale dual map $\beta^\vee : \mathbb{Z}^m \to DG(\beta)$ in (2.2). We denote the Gale dual map of

$$\mathbb{Z}^m \oplus \mathbb{Z}^m \xrightarrow{(\beta^\vee, -\beta^\vee)} DG(\beta)$$

by

$$\beta_L : \mathbb{Z}^{2m} \to N_L, \tag{2.4}$$

where $\overline{N}_L$ is a lattice of dimension $2m - (m - d)$. The map β_L is given by the integral vectors $\{b_{L,1}, \cdots, b_{L,m}, b'_{L,1}, \cdots, b'_{L,m}\}$ and β_L is called the Lawrence lifting of β.

Remark 2.3. By [28, Remark 2.3], the lattice $N_L = N \oplus \mathbb{Z}^m$ and $\{b_{L,1}, \cdots, b_{L,m}, b'_{L,1}, \cdots, b'_{L,m}\}$ are the vectors

$$\{(b_1, e_1), \cdots, (b_m, e_m), (0, e_1), \cdots, (0, e_m)\},$$

where $\{e_i\}$ are the standard bases of $\mathbb{Z}^m$.

Given the generic element θ, let $\overline{\theta}$ be the natural image of θ under the projection $DG(\beta) \to \overline{DG(\beta)}$. Then the map $\overline{\beta}^\vee : \mathbb{Z}^m \to \overline{DG(\beta)}$ is given by $\overline{\beta}^\vee = (\overline{a}_1, \cdots, \overline{a}_m)$. For any basis of $\overline{DG(\beta)}$ of the form $C = \{\overline{a}_{i_1}, \cdots, \overline{a}_{i_{m-d}}\}$, there exist unique $\lambda_1, \cdots, \lambda_{m-d}$ such that

$$\overline{a}_{i_1}\lambda_1 + \cdots + \overline{a}_{i_{m-d}}\lambda_{m-d} = \overline{\theta}.$$

Let $\mathbb{C}[z_1, \cdots, z_m, w_1, \cdots, w_m]$ be the coordinate ring of $\mathbb{C}^{2m}$. Let

$$\sigma(C, \theta) = \{\overline{b}_{L,i_j} \mid \lambda_j > 0\} \sqcup \{\overline{b}'_{L,i_j} \mid \lambda_j < 0\} \quad \text{and} \quad C(\theta) = \{z_{i_j} \mid \lambda_j > 0\} \sqcup \{w_{i_j} \mid \lambda_j < 0\}.$$

We put

$$\mathcal{I}_\theta := \left\langle \prod C(\theta) \mid C \text{ is a basis of } \overline{DG(\beta)} \right\rangle, \tag{2.5}$$

and

$$\Sigma_\theta := \{\overline{\sigma}(C, \theta) : \ C \text{ is a basis of } \overline{DG(\beta)}\}, \tag{2.6}$$

where $\overline{\sigma}(C, \theta) = \{\overline{b}_{L,1}, \cdots, \overline{b}_{L,m}, \overline{b}'_{L,1}, \cdots, \overline{b}'_{L,m}\} \setminus \sigma(C, \theta)$ is the complement of $\sigma(C, \theta)$ and corresponds to a maximal cone in Σ_θ. From [23], Σ_θ is the fan of a Lawrence toric variety $X(\Sigma_\theta)$ corresponding to θ in the lattice $\overline{N}_L$, and $\mathcal{I}_\theta$ is the irrelevant ideal. The construction above establishes the following

Proposition 2.4 ([28], Proposition 2.5)**.** *A stacky hyperplane arrangement $\mathcal{A} = (N, \beta, \theta)$ also gives a stacky fan $\mathbf{\Sigma}_\theta = (N_L, \Sigma_\theta, \beta_L)$ which is called a Lawrence stacky fan.*

Definition 2.5 ([28], Definition 2.6)**.** The toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}_\theta)$ is called the Lawrence toric Deligne-Mumford stack.

For the map $\beta_L^\vee : \mathbb{Z}^m \oplus \mathbb{Z}^m \to DG(\beta)$ given by $(\beta^\vee, -\beta^\vee)$, there is an exact sequence

$$0 \longrightarrow N_L^* \longrightarrow \mathbb{Z}^{2m} \xrightarrow{\beta_L^\vee} DG(\beta) \longrightarrow Coker(\beta_L^\vee) \longrightarrow 0. \tag{2.7}$$

Applying $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to (2.7) yields

$$1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha^L} (\mathbb{C}^\times)^{2m} \longrightarrow T_L \longrightarrow 1, \tag{2.8}$$

where $\mu := Hom_{\mathbb{Z}}(Coker(\beta_L^\vee), \mathbb{C}^\times)$ and T_L is the torus of dimension $m + d$. From [6] and Proposition 2.4, the toric DM stack $\mathcal{X}(\mathbf{\Sigma}_\theta)$ is the quotient stack $[(\mathbb{C}^{2m} \setminus V(\mathcal{I}_\theta))/G]$, where G acts on $\mathbb{C}^{2m} \setminus V(\mathcal{I}_\theta)$ through the map α^L in (2.8).

2.1.3. *Hypertoric Deligne-Mumford stacks.* Define an ideal in $\mathbb{C}[z, w]$ by:

$$I_{\beta^\vee} := \left\langle \sum_{i=1}^{m} (\beta^\vee)^*(x)_i z_i w_i \mid x \in DG(\beta)^* \right\rangle, \tag{2.9}$$

where $(\beta^\vee)^*$ is the map in (2.1) and $(\beta^\vee)^*(x)_i$ is the i-th component of the vector $(\beta^\vee)^*(x)$.

According to Section 6 in [23], $I_{\beta^\vee}$ is a prime ideal. Let Y be the closed subvariety of $\mathbb{C}^{2m} \setminus V(\mathcal{I}_\theta)$ determined by the ideal (2.9). Since $(\mathbb{C}^\times)^{2m}$ acts on Y naturally and the group G acts on Y through the map α^L, we have the quotient stack $[Y/G]$. Since $Y \subseteq \mathbb{C}^{2m} \setminus V(\mathcal{I}_\theta)$ is a closed subvariety, the quotient stack $[Y/G]$ is a closed substack of $\mathcal{X}(\mathbf{\Sigma}_\theta)$, and is Deligne-Mumford.

Definition 2.6 ([28], Definition 2.7)**.** The hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ associated to the stacky hyperplane arrangement $\mathcal{A}$ is defined to be the quotient stack $[Y/G]$.

Example 2.7. Let $N = \mathbb{Z}$, Σ the fan of projective line $\mathbb{P}^1$, and $\beta : \mathbb{Z}^2 \to N$ given by $\{b_1 = (-2), b_2 = (1)\}$. Then the Gale dual $\beta^\vee : \mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $\begin{bmatrix} 1, & 2 \end{bmatrix}$. Choose a generic element $\theta = (1)$ in $\mathbb{Z}$ which determines the fan Σ. The stacky hyperplane arrangement is $\mathcal{A} = (N, \beta, \theta)$, $G = (\mathbb{C}^\times)^1$ and Y is the subvariety of $\operatorname{Spec}(\mathbb{C}[z_1, z_2, w_1, w_2])$ determined by the ideal $I_{\beta^\vee} = (z_1 w_1 + z_2 w_2)$. Then the hypertoric Deligne-Mumford stack is the cotangent bundle $T^*_{\mathbb{P}(1,2)}$.

Each Deligne-Mumford stack has an underlying coarse moduli space. Consider again the map $\beta^\vee : \mathbb{Z}^m \to DG(\beta)$ in (2.2), which is given by the vectors $(a_1, \cdots, a_m)$. As in §2.1.2, let $\overline{\theta}$ be the natural image of θ under the projection $DG(\beta) \to \overline{DG(\beta)}$. Then the map $\overline{\beta}^\vee : \mathbb{Z}^m \to \overline{DG(\beta)}$ is given by $\overline{\beta}^\vee = (\overline{a}_1, \cdots, \overline{a}_m)$. From the map $\overline{\beta}^\vee$ we get the simplicial fan Σ_θ in (2.6). By [6], the toric variety $X(\Sigma_\theta) = (\mathbb{C}^{2m} - V(\mathcal{I}_\theta))/G$, is the coarse moduli space of the Lawrence toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}_\theta)$. The toric variety $X(\Sigma_\theta)$ is semi-projective, but not projective. In [23], from $\beta^\vee$ and θ, the authors define the hypertoric variety $Y(\beta^\vee, \theta)$ as the complete intersection of the toric variety $X(\Sigma_\theta)$ by the ideal (2.9), which is the geometric quotient Y/G.

Proposition 2.8 ([28], Proposition 2.8)**.** *The coarse moduli space of $X_{\mathcal{A}}$ is $Y(\beta^\vee, \theta)$.*

2.1.4. *Cores.* Recall that a hypertoric Deligne-Mumford stack $X_{\mathcal{A}} \to \overline{X}_0$ is a symplectic resolution, and the core $C(X_{\mathcal{A}})$ is the fibre over most singular points of $\overline{X}_0$, which is the deformation retract of $X_{\mathcal{A}}$.

The core is a finite union of toric Deligne-Mumford stacks. Let $U \subset [m]$ be a finite subset, and set

$$\mathcal{P}_U := \{v \in M_{\mathbb{R}} | \langle b_i, v\rangle + r_i \geq 0 \text{ if } i \in U, \text{ and } \langle b_i, v\rangle + r_i \leq 0 \text{ if } i \notin U\}.$$

Then $\mathcal{P}_U$ is the polytope cut out by the cooriented hyperplanes of $\mathcal{H} = \{H_1, \cdots, H_m\}$ after reversing the coorientations of the hyperplanes with indices in U. Assume that $\mathcal{P}_U$ is bounded, we denote by $\Sigma_U \subset N_{\mathbb{R}}$ the normal fan of $\mathcal{P}_U$. Then $\mathbf{\Sigma}_U = (N, \Sigma_U, \beta)$ is an extended stacky fan in sense of [27], and let $\chi(\mathbf{\Sigma_U})$ be the corresponding toric Deligne-Mumford stack.

Proposition 2.9. *The core $C(X_{\mathcal{A}})$ of hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is*

$$C(X_{\mathcal{A}}) = \bigcup_{\mathcal{P}_U \ \textit{bounded}} \chi(\mathbf{\Sigma_U}).$$

Example 2.10. Let (N, β, θ) be a stacky hyperplane arrangementgiven by:

$$\mathbb{Z}^4 \xrightarrow{\beta} \mathbb{Z}^2,$$

and β is given by

$$\begin{cases} b_1 = (1,0); \\ b_2 = (0,-1); \\ b_3 = (0,1); \\ b_4 = (-1,-2). \end{cases}$$

The generic element $\theta = (1,1) \in DG(\beta) = \mathbb{Z}^2$. The normal fan of the bounded polytope Γ is the toric fan of a Hirzebruch surface. The core of the hypertoric Deligne-Mumford stack is a union of Hirzebruch surface and a weighted projective plane $\mathbb{P}(1,1,2)$.

2.1.5. *Closed substacks of hypertoric Deligne-Mumford stacks.* Let $\mathcal{A} = (N, \beta, \theta)$ be a stacky hyperplane arrangement. Consider the map $\beta : \mathbb{Z}^m \to N$ given by $\{b_1, \cdots, b_m\}$. Let Cone(β) be a partially ordered finite set of cones generated by $\overline{b}_1, \cdots, \overline{b}_m$. The partial ordering is defined by requiring that $\sigma \prec \tau$ if σ is a face of τ. We have the minimum element $\hat{0}$ which is the cone consisting of the origin. Let Cone($\overline{N}$) be the set of all convex polyhedral cones in the lattice $\overline{N}$. Then we have a map

$$C : \mathrm{Cone}(\beta) \longrightarrow \mathrm{Cone}(\overline{N}),$$

such that for any $\sigma \in \mathrm{Cone}(\beta)$, $C(\sigma)$ is the cone in $\overline{N}$. Then $\Delta_\beta := (C, \mathrm{Cone}(\beta))$ is a simplicial multi-fan in the sense of [25].

Let $\sigma \in \Delta_\beta$ be a cone. According to [28, Section 4], the stacky hyperplane arrangement $\mathcal{A} = (N, \beta, \theta)$ induces a quotient stacky hyperplane arrangement $\mathcal{A}/\sigma = (N(\sigma), \beta(\sigma), \theta(\sigma))$, whose corresponding hypertoric Deligne-Mumford stack $X_{\mathcal{A}/\sigma}$ is a closed substack of $X_{\mathcal{A}}$. More details can be found in [28, Section 4].

Example 2.11. [Inertia stacks] Let N_σ be the sublattice generated by σ, and $N(\sigma) := N/N_\sigma$. Note that when σ is a top dimensional cone, $N(\sigma)$ is the local orbifold group in the local chart of the coarse moduli space of the hypertoric toric Deligne-Mumford stack. Given the multi-fan Δ_β, we consider the pairs (v, σ), where σ is a cone in Δ_β, $v \in N$ such that $\overline{v} = \sum_{\rho_i \subseteq \sigma} \alpha_i b_i$ for $0 < \alpha_i < 1$. Note that σ is the minimal cone in Δ_β satisfying the above condition. Let Box := Box(Δ_β) be the set of all such pairs (v, σ).

[28, Proposition 4.7] determines the inertia stack of $X_{\mathcal{A}}$, which is given by

$$IX_{\mathcal{A}} = \coprod_{(v,\sigma) \in \mathrm{Box}(\Delta_\beta)} X_{\mathcal{A}/\sigma}. \tag{2.10}$$

2.2. **Equivariant Chen-Ruan cohomology.** Let $X_{\mathcal{A}}$ be the hypertoric Deligne-Mumford stack associated to a stacky hyperplane arrangement $\mathcal{A}$. From the construction in §2.1.3, there is a torus $T := (\mathbb{C}^\times)^m$ action on $X_{\mathcal{A}}$. From the exact sequence:

$$1 \to \mu \longrightarrow G \longrightarrow (\mathbb{C}^\times)^m \longrightarrow (\mathbb{C}^\times)^d \to 1$$

the T-action on $X_{\mathcal{A}}$ induces a $(\mathbb{C}^\times)^d$-action on $X_{\mathcal{A}}$.

There is another $\mathbb{C}^\times$-action on the fibre of

$$T^*\mathbb{C}^m = \mathbb{C}^m \times (\mathbb{C}^m)^*$$

by scaling. We consider $\mathbb{T} := T \times \mathbb{C}^\times$-equivariant Chen-Ruan cohomology of $X_{\mathcal{A}}$.

2.2.1. $\mathbb{T}$-*equivariant cohomology of* $X_{\mathcal{A}}$. The torus T-equivariant cohomology of $X_{\mathcal{A}}$ is described by the multi-fan Δ_β, i.e. the Matroid M_β. Each $b_i \in \mathcal{A}$ defines a line bundle

$$L_i = [Y \times \mathbb{C}/G]$$

where G acts on the fibre by the i-th component $\alpha : G \to (\mathbb{C}^\times)^m$. Let λ_i be the parameters of the T-action on $X_{\mathcal{A}}$. Let $\{u_i | 1 \leq i \leq m\}$ be the T-equivariant Chern class of L_i over $X_{\mathcal{A}}$. If D_i is the divisor corresponding to the line bundle L_i. Then

$$u_i = D_i - \lambda_i.$$

The following result is due to Hausel and Sturmfel [23]:

Proposition 2.12. *Let $\mathcal{A} = (N, \beta, \theta)$ be a stacky hyperplane arrangement and $X_{\mathcal{A}}$ the corresponding hypertoric Deligne-Mumford stack. Then*

$$H_T^*(X_{\mathcal{A}}) = R_{\mathbb{T}}[u_1, \cdots, u_m]/I_{M_\beta},$$

where

$$I_{M_\beta} = \{u_{i_1} \cdots u_{i_k} | \overline{b}_{i_1}, \cdots, \overline{b}_{i_k} \text{ linearly dependent in } \overline{N}\}.$$

We consider the extra factor $\mathbb{C}^\times$ action in the $\mathbb{T}$-equivariant cohoology of $X_{\mathcal{A}}$. The extra $\mathbb{C}^\times$-action on $T^*\mathbb{C}^m$ descends to a $\mathbb{T}$-equivariant line bundle over $X_{\mathcal{A}}$ with the first Chern class $\hbar$. Recall the line bundle L_i over $X_{\mathcal{A}}$ just constructed before. The line bundle L_i can be thought as a divisor

$$\{(z_i, w_i) \in Y | z_i = 0\}.$$

Now if we work $\mathbb{T}$-equivariantly, we will have the following L_i^{-1} over $X_{\mathcal{A}}$ as:

$$\{(z_i, w_i) \in Y | w_i = 0\}$$

and we have a $\mathbb{C}^\times$-action on L_i^{-1}, so

$$c_1(L_i^{-1}) = \hbar - u_i.$$

As in [21], [24], define

$$G_i := \{v \in M_{\mathbb{R}} | < b_i, v > + r_i \leq 0\}.$$

Definition 2.13. A circuit $S \subset \mathcal{A}$ is a minimal subset of hyperplanes satisfying $\cap_{i \in S} H_i = \emptyset$, and let $S := S^+ \sqcup S^-$ be the unique splitting such that

$$(\cap_{i \in S^+} G_i) \cap (\cap_{j \in S^-} F_j) = \emptyset.$$

Let $S = S^+ \sqcup S^-$ be a minimal circuit, so that $\{b_{i_k} | i_k \in S\}$ linearly dependent in $\overline{N}$. Then there exists positive integers $w_i \in \mathbb{Z}_{>0}$ for $i \in S$ such that

$$\sum_{i \in S^+} w_i \overline{b}_i - \sum_{j \in S^-} w_j \overline{b}_j = 0.$$

Let

$$\beta_S = \sum_{i \in S^+} w_i e_i - \sum_{j \in S^-} w_j e_j,$$

then by exact sequence (2.1) β_S determines an element in $DG(\beta)^* = H_2(X_{\mathcal{A}})$.

Example 2.14. Let the stacky hyperplane arrangement (N, β, θ) be given by:

$$\mathbb{Z}^4 \xrightarrow{\beta} \mathbb{Z}^2,$$

and β is by

$$\begin{cases} b_1 = (1,0); \\ b_2 = (0,-1); \\ b_3 = (0,1); \\ b_4 = (-1,-1). \end{cases}$$

The generic element $\theta = (1,1) \in DG(\beta) = \mathbb{Z}^2$. The normal fan of the bounded polytope Γ is the toric fan of a Hirzebruch surface. The core of the hypertoric Deligne-Mumford stack is a union of Hirzebruch surface and a projective plane. We have three circuits in this case:

$$S = (1,2,4); \quad S = (1,3,4); \quad S = (2,3).$$

(1) $S = S^+ \sqcup S^- = (1,4) \sqcup (2)$. Then $\beta_S = e_1 + e_4 - e_2 = (1,-1,0,1)$. Then it determines an element $(0,1) \in \mathbb{Z}^2$;
(2) $S = S^+ = (1,3,4)$. Then $\beta_S = e_1 + e_3 + e_4 = (1,0,1,1)$. Then it determines an element $(1,1) \in \mathbb{Z}^2$;
(3) $S = S^+ = (2,3)$. Then $\beta_S = e_2 + e_3 = (0,1,1,0)$. Then it determines an element $(1,0) \in \mathbb{Z}^2$.

Example 2.15. Look at Example 2.10 again. We have three circuits in this case:

$$S = (1,2,4); \quad S = (1,3,4); \quad S = (2,3).$$

(1) $S = S^+ \sqcup S^- = (1,4) \sqcup (2)$. Then $b_1 + b_4 - 2b_2 = 0$ and $\beta_S = e_1 + e_4 - 2e_2 = (1,-2,0,1)$. Then it determines an element $(0,1) \in \mathbb{Z}^2$;
(2) $S = S^+ = (1,3,4)$. Then $b_1 + 2b_3 + b_4 = 0$ and $\beta_S = e_1 + 2e_3 + e_4 = (1,0,1,1)$. Then it determines an element $(2,1) \in \mathbb{Z}^2$;
(3) $S = S^+ = (2,3)$. Then $b_2 + b_3 = 0$ and $\beta_S = e_2 + e_3 = (0,1,1,0)$. Then it determines an element $(1,0) \in \mathbb{Z}^2$.

Theorem 2.16. *The $\mathbb{T}$-equivariant cohomology ring of $X_{\mathcal{A}}$ is given by:*

$$H^*_{\mathbb{T}}(X_{\mathcal{A}}) \cong \frac{R_{\mathbb{T}}[u_1, \cdots, u_m, \hbar]}{\{\langle \prod_{i \in S^+} u_i \cdot \prod_{j \in S^-} (\hbar - u_j) \rangle | S \textit{ a circuit}\}}.$$

Proof. Recall that $X_{\mathcal{A}}$ is a closed substack of the Lawrence toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}_\theta)$. From the main result in [29], the cohomology of $X_{\mathcal{A}}$ is isomorphic to the cohomology ring of $\mathcal{X}(\mathbf{\Sigma}_\theta)$. There is a torus $\mathbb{T}$ action on $\mathcal{X}(\mathbf{\Sigma}_\theta)$ and the $\mathbb{T}$-equivariant cohomology ring of $X_{\mathcal{A}}$ is isomorphic to the $\mathbb{T}$-equivariant cohomology ring of $\mathcal{X}(\mathbf{\Sigma}_\theta)$. We have:

$$H^*_{\mathbb{T}}(\mathcal{X}(\mathbf{\Sigma}_\theta)) \cong \frac{R_{\mathbb{T}}[u_1, \cdots, u_m, v_1, \cdots, v_m]}{I_\theta},$$

where

$$I_\theta = \bigcap_C \langle \sigma(C, \theta) \rangle \subset \mathbb{Q}[u, v]$$

and $C = \{a_{i_1}, \cdots, a_{i_d}\}$, $\lambda_1 a_{i_1} + \cdots + \lambda_d a_{i_d} = \theta$ so that

$$\sigma(C, \theta) = \{u_{i_j} : \lambda_j > 0\} \cup \{u_{i_j} : \lambda_j < 0\}.$$

Now we put into the extra $\mathbb{C}^\times$-action on the fibre of $T^*\mathbb{C}^m$. The line bundle L_i^{-1} over $\mathcal{X}(\mathbf{\Sigma}_\theta)$ will admit such a $\mathbb{C}^\times$ action. Look at the ideal

$$I_\theta = \bigcap \langle C, \theta\rangle,$$

as in [23, Section 4], I_θ corresponds to all

$$\{S = (i_1, \cdots, i_k) | \overline{b}_{i_1}, \cdots, \overline{b}_{i_k} \text{ linearly dependent in } \overline{N}\}.$$

Now let $S = S^+ \sqcup S^-$ be the decomposition such that $i_j \in S^+$ if $\lambda_j > 0$ and $i_j \in S^-$ if $\lambda_j < 0$. Note that in equivariant setting we have $v_i = \hbar - u_i$. So the formula in the theorem just follows. □

2.2.2. $\mathbb{T}$-*equivariant Chen-Ruan cohomology.* The Chen-Ruan cohomology of $X_{\mathcal{A}}$ was calculated algebraically in [28] and symplectically in [21]. Here we calculate the $\mathbb{T}$-*equivariant* Chen-Ruan cohomology of $X_{\mathcal{A}}$.

Recall that the twisted sectors (components of the inertia stack $IX_{\mathcal{A}}$) of the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ are given by the box elements $\mathrm{Box}(\Delta_\beta)$ of the multi-fan Δ_β. The set $\mathrm{Box}(\Delta_\beta)$ consists of all pairs (v, σ), where σ is a cone in the multi-fan Δ_β, $v \in N$ such that $\overline{v} = \sum_{\rho_i \subseteq \sigma} \alpha_i \overline{b}_i$ for $0 < \alpha_i < 1$. For $(v, \sigma) \in \mathrm{Box}(\Delta_\beta)$ we consider a closed substack of $X_{\mathcal{A}}$ given by the quotient stacky hyperplane arrangement $\mathcal{A}(\sigma)$. The inertia stack of $X_{\mathcal{A}}$ is the disjoint union of all such closed substacks, see Example 2.11 or [28, Section 4].

We introduce a variable $\mathbb{1}_{(v,\sigma)}$ for each box element (v, σ). We make a convention that $\mathbb{1}_{(v,\sigma)} = 1$ if $(v, \sigma) = (0, 0)$ is the trivial box element. Let $(v, \sigma) \in \mathrm{Box}(\Delta_\beta)$, say $v \in N(\tau)$ for some top dimensional cone τ. Let $(\check{v}, \sigma) \in \mathrm{Box}(\Delta_\beta)$ be the inverse of v as an element in the group $N(\tau)$. Equivalently, if $\overline{v} = \sum_{\rho_i \subseteq \sigma} \alpha_i \overline{b}_i$ for $0 < \alpha_i < 1$, then $\check{\overline{v}} = \sum_{\rho_i \subseteq \sigma} (1 - \alpha_i) \overline{b}_i$.

For any two box elements $(v_1, \tau_1), (v_2, \tau_2) \in \mathrm{Box}(\Delta_\beta)$, if $\tau_1 \cup \tau_2$ is a cone in Δ_β, then there is a (v_3, σ_3) which is unique in $\mathrm{Box}(\Delta_\beta)$ such that $v_1 + v_2 + v_3 \equiv 0$ in the local group given by $\sigma_1 \cup \sigma_2$. Let $\overline{v}_i = \sum_{\rho_j \subseteq \sigma_i} \alpha_j^i \overline{b}_j$, with $0 < \alpha_j^i < 1$ and $i = 1, 2, 3$. The existence of $\alpha_j^1, \alpha_j^2, \alpha_j^3$ means that ρ_j belongs to $\sigma_1, \sigma_2, \sigma_3$. Let σ_{123} be the cone in Δ_β such that $\overline{v}_1 + \overline{v}_2 + \overline{v}_3 = \sum_{\rho_i \subseteq \sigma_{123}} a_i \overline{b}_i$, with $a_i = 1$ or 2. Let I be the set of i such that $a_i = 1$ and $\alpha_j^1, \alpha_j^2, \alpha_j^3$ exist, J the set of j such that ρ_j belongs to σ_{123} but not to σ_3.

Theorem 2.17. *The $\mathbb{T}$-equivariant Chen-Ruan cohomology $H^*_{CR,\mathbb{T}}(X_{\mathcal{A}})$ is given by*

$$H^*_{CR,\mathbb{T}}(X_{\mathcal{A}}) = \frac{R_{\mathbb{T}}[u_1, \cdots, u_m, \hbar, \{\mathbb{1}_{(v,\sigma)}\}_{(v,\sigma) \in Box(\Delta_\beta)}]}{I + J},$$

where

(1) I is the ideal of all products $\langle \prod_{i \in S^+} u_i \cdot \prod_{j \in S^-} (\hbar - u_j) \rangle$ for all circuits $S \subset \mathcal{A}$;
(2) J is the ideal generated by the relations:

(2.11)
$$\begin{cases} \mathbb{1}_{(v,\tau)} \cdot u_i = 0, & \tau \cup \rho_i \text{ is not a cone in } \Delta_\beta; \\ \mathbb{1}_{(v_1,\tau_1)} \cdot \mathbb{1}_{(v_2,\tau_2)} = (-1)^{|I|+|J|} \mathbb{1}_{(v_3,\tau_3)} \cdot \prod_{i \in I} u_i \cdot \prod_{i \in J} u_i^2, & \tau \cup \rho_i \text{ is a cone in } \Delta_\beta \text{ and } \overline{v}_1 \neq \check{\overline{v}}_2; \\ \mathbb{1}_{(v_1,\tau_1)} \cdot \mathbb{1}_{(v_2,\tau_2)} = (-1)^{|J|} \cdot \prod_{i \in J} u_i^2, & \tau \cup \rho_i \text{ is a cone in } \Delta_\beta \text{ and } \overline{v}_1 = \check{\overline{v}}_2; \\ \mathbb{1}_{(v_1,\tau_1)} \cdot \mathbb{1}_{(v_2,\tau_2)} = 0, & \tau \cup \rho_i \text{ is not a cone in } \Delta_\beta. \end{cases}$$

Proof. Let M_β be the matroid determined by β. Then $\mathbb{Q}[M_\beta]$ is the cohomology ring of hypertoric Deligne-Mumford stack $\mathcal{M}(\mathcal{A})$ and we write:

$$H^*_{\mathbb{T}}(X_{\mathcal{A}}) \cong R_{\mathbb{T}}[M_\beta] = \frac{R_{\mathbb{T}}[u_1, \cdots, u_m, \hbar]}{I_{M_\beta}}.$$

By the definition of Chen-Ruan cohomology

$$H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) = \bigoplus_{(v,\sigma)\in \mathrm{Box}(\Delta_\beta)} H^*_{\mathbb{T}}(X_{\mathcal{A}/\sigma}),$$

where $H^*_{\mathbb{T}}(X_{\mathcal{A}/\sigma})$ is the $\mathbb{T}$-equivariant cohomology of the twisted sector $X_{\mathcal{A}/\sigma}$. According to [28], $H^*_{\mathbb{T}}(X_{\mathcal{A}/\sigma})$ is isomorphic to $R_{\mathbb{T}}[M_{\beta(\sigma)}]$, the equivariant cohomology ring of the induced matroid $M_{\beta(\sigma)}$. By the proof of [28, Proposition 5.8], $R_{\mathbb{T}}[M_{\beta(\sigma)}] \cong \mathbb{1}_{(v,\sigma)} \cdot R_{\mathbb{T}}[M_\beta]$. Then as a vector space, we have

$$H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) = \bigoplus_{(v,\sigma)\in \mathrm{Box}(\Delta_\beta)} \mathbb{1}_{(v,\sigma)} \cdot R_{\mathbb{T}}[M_\beta].$$

Then to prove the formula in the theorem, it is sufficient to prove the orbifold cup product of the box elements and the generators u_i satisfy the relation in (2), but all of these relations come from the definition of orbifold cup product calculated in [28, Lemma 5.13] and in the proof of [28, Theorem 1.1] in [28, Section 5.3]. □

2.3. **Gromov-Witten theory.** We briefly review some basic definitions on orbifold Gromov-Witten theory and the reduced virtual fundamental cycles.

2.3.1. *Equivariant Gromov-Witten invariants.* Let X be a smooth Deligne-Mumford stack, endowed with a torus T action. The moduli stack $\overline{\mathcal{M}}_{0,n}(X,d)$ of degree $d \in H_2(X,\mathbb{Q})$ twisted stable maps to X carries a T-action, and a virtual fundamental cycle $[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{virt}} \in A_{*,T}(\overline{\mathcal{M}}_{0,n}(X,d))$. There are T-equivariant evaluation maps[2]:

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to IX$$

to the inertia stack IX of X for $1 \leq i \leq n$, see [11], [1].

Given $\gamma_1, ..., \gamma_n \in H^*_{\mathrm{CR},T}(X)$, we consider the following genus 0 T-equivariant Gromov-Witten invariant:

$$\langle \gamma_1, \cdots, \gamma_n \rangle^X_{0,n,d} = \int^T_{[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{virt}}} \prod_i \mathrm{ev}_i^\star \gamma_i.$$

The moduli stack $\overline{\mathcal{M}}_{0,n}(X,d)$ has components indexed by the components of inertia stack IX. We write

$$IX = \bigsqcup_{f\in \mathsf{B}} X_f$$

for the decomposition of IX into connected components, where B is the index set. Then the component $\overline{\mathcal{M}}_{0,n}(X,d)^{f_1,\cdots,f_n}$ is the one which under evaluation maps ev_i, the images lie in the component X_{f_i}. The virtual dimension of $\overline{\mathcal{M}}_{0,n}(X,d)^{f_1,\cdots,f_n}$ is:

$$-K_X \cdot d + \dim(X) + n - 3 - \sum_i \mathrm{age}(X_{f_i}). \tag{2.12}$$

[2]We ignore the issue of trivializing the marked gerbes in our moduli problem. A detailed discussion on this can be found in [1].

If X is not compact (like our hypertoric Deligne-Mumford stacks), then the moduli stack $\overline{\mathcal{M}}_{0,n}(X,d)$ is non-compact. There is a T-action on $\overline{\mathcal{M}}_{0,n}(X,d)$. Assume that the T-fixed locus $\overline{\mathcal{M}}_{0,n}(X,d)^T$ is compact, then T-equivariant GW invariants can be defined in the same way, replacing equivariant integration by equivariant residues.

Let $\mathrm{NE}(X) \subset H_2(X,\mathbb{R})$ be the cone generated by classes of effective curves and set

$$\mathrm{NE}(X)_{\mathbb{Z}} := \{d \in H_2(X,\mathbb{Z}) : d \in \mathrm{NE}(X)\}.$$

Let $R_T := H_T^*(pt)$ and $R_T[\![Q]\!]$ the formal power series ring

$$R_T[\![Q]\!] = \left\{ \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} a_d Q^d : a_d \in R \right\}$$

so that Q is a so-called *Novikov variable* [37, III 5.2.1]. For $\gamma_i, \gamma_j, t \in H^*_{\mathrm{CR},T}(X)$, the big $\mathbb{T}$-equivariant quantum product is defined by:

$$(\gamma_i \star_{\mathrm{big},t} \gamma_j, \gamma_k) = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} \sum_{n \geq 0} Q^d \langle \gamma_i, \gamma_j, \underbrace{t, \cdots, t}_{n}, \gamma_k \rangle^X_{0,n+3,d} \tag{2.13}$$

The small $\mathbb{T}$-equivariant quantum product is defined by putting $n = 0$:

$$(\gamma_i \star \gamma_j, \gamma_k) = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} Q^d \langle \gamma_i, \gamma_j, \gamma_k \rangle^X_{0,3,d} \tag{2.14}$$

or

$$\gamma_i \star \gamma_j = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} Q^d \cdot \mathrm{inv}^\star \cdot \mathrm{ev}_{3,\star}(\mathrm{ev}_1^\star(\gamma_i)\, \mathrm{ev}_j^\star(\gamma_j) \cap [\overline{\mathcal{M}}_{0,3}(X,d)]^{\mathrm{virt}})$$

where $\mathrm{inv} : IX \to IX$ denotes the involution sending $(x,g) \mapsto (x,g^{-1})$, for $x \in X, g \in \mathrm{Aut}(x)$. The quantum product satisfies the associativity property and makes $H^*_{\mathrm{CR},T}(X) \otimes R_T[\![Q]\!]$ a ring, which is called the small equivariant quantum cohomology ring.

Assume that $d \neq 0$ and $D \in H^2(X,\mathbb{Q})$ a divisor class. Then the *divisor equation* of Gromov-Witten invariants is:

$$\langle D, \gamma_i, \gamma_j \rangle^X_{0,3,d} = (D \cdot d) \cdot \langle \gamma_i, \gamma_j \rangle^X_{0,2,d}. \tag{2.15}$$

So in order to calculate the quantum product by a divisor D, it suffices to study two point Gromov-Witten invariants of X.

2.3.2. *Reduced virtual fundamental cycle.* Let X be a smooth quasi-projective Deligne-Mumford stack with a nowhere vanishing holomorphic symplectic form $\omega \in \Omega^{0,2}(X)$. By [32], the usual non-equivariant virtual fundamental cycle vanishes when $d \neq 0$. The hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ admits such a holomorphic symplectic form ω induced from the canonical symplectic form on $T^*\mathbb{C}^m$. In this situation we need to consider reduced virtual fundamental classes in order to obtain non-trivial invariants.

We recall the reduced virtual fundamental cycle construction following [7, Section 4.2] (this is a special case of cosection localized cycle [32]). Let $\mathcal{C}$ be a fixed twisted nodal curve, and let $\overline{\mathcal{M}}_{\mathcal{C}} := \overline{\mathcal{M}}_{\mathcal{C}}(X,d)$ denote the moduli stack of twisted maps from $\mathcal{C}$ to X with degree d. Recall that in [4], the obstruction theory for $\overline{\mathcal{M}}_{\mathcal{C}}$ is given by:

$$R\pi_\star(\mathrm{ev}^\star T_X)^\vee \to L_{\overline{\mathcal{M}}_{\mathcal{C}}},$$

where $L_{\overline{\mathcal{M}}_{\mathcal{C}}}$ is the cotangent complex of $\overline{\mathcal{M}}_{\mathcal{C}}$, and

$$\mathrm{ev} : \mathcal{C} \times \overline{\mathcal{M}}_{\mathcal{C}} \to IX$$

$$\pi : \mathcal{C} \times \overline{\mathcal{M}}_{\mathcal{C}} \to \overline{\mathcal{M}}_{\mathcal{C}}$$

are evaluation maps and projection to $\overline{\mathcal{M}}_{\mathcal{C}}$, respectively.

Denote by ω_π the relative dualizing sheaf. Pairing with the symplectic form and pullback differentials we have a map:

$$\mathrm{ev}^\star(T_X) \to \omega_\pi \otimes (\mathbb{C}\omega)^*.$$

Then this induces a morphism

$$R\pi_\star(\omega_\pi)^\vee \otimes \mathbb{C}\omega \to R\pi_\star(\mathrm{ev}^\star(T_X)^\vee).$$

Taking truncation we get a morphism:

$$\iota : \tau_{\leq -1} R\pi_\star(\omega_\pi)^\vee \otimes \mathbb{C}\omega \to R\pi_\star(\mathrm{ev}^\star(T_X)^\vee).$$

The truncation is a trivial line bundle, but carries a nontrivial action in the equivariant setting depending on the T-action on X.

There is an induced map from the mapping cone of ι:

$$C(\iota) \to L_{\overline{\mathcal{M}}_{\mathcal{C}}} \tag{2.16}$$

which satisfies the conditions in the perfect obstruction theory of [3], [35]. The reduced virtual fundamental cycle of $\overline{\mathcal{M}}_{\mathcal{C}}$ is defined by the reduced obstruction theory (2.16). The reduced virtual fundamental class $[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{red}}$ for $\overline{\mathcal{M}}_{0,n}(X,d)$ is obtained by applying the construction above to the universal family over $\overline{\mathcal{M}}_{0,n}(X,d)$.

In the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ case, there is a $\mathbb{T}$-action on $X_{\mathcal{A}}$ and the moduli stack $\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}},d)$. The extra $\mathbb{C}^\times$ acts on the space of symplectic forms $\mathbb{C}\omega$ with nontrivial weights. Hence from (2.16),

$$[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}},d)]^{\mathrm{virt}} = \hbar[\overline{\mathcal{M}}_{0,n}(X_{\mathcal{A}},d)]^{\mathrm{red}}. \tag{2.17}$$

The detail argument of this relation can be found in [7, Section 4.2]. Although [7] works with smooth schemes, the arguments hold true for smooth Deligne-Mumford stacks.

2.3.3. *Maps from $\mathbb{P}^1_{s_1,s_2}$ to $X_{\mathcal{A}}$.* Let $\mathbb{P}^1_{s_1,s_2}$ be the unique $\mathbb{P}^1$-orbifold with stacky points $P_1 = [1,0] = B\mu_{s_1}$ and $P_2 = [0,1] = B\mu_{s_2}$, and no other stacky points. For the purpose of calculation, we classify the morphisms from $\mathbb{P}^1_{s_1,s_2}$ to $X_{\mathcal{A}}$.

Recall that our hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is open, and the core is a union of toric Deligne-Mumford stacks given by the bounded polytope of the stacky hyperplane arrangement, see §2.1.4. A morphism $\mathbb{P}^1_{s_1,s_2} \to X_{\mathcal{A}}$ must have image in an irreducible component of the core, i.e. must lie in a toric Deligne-Mumford stack inside the core. So an argument similar to [14, Section 3] works for hypertoric Deligne-mumford stacks.

The torus $\mathbb{T}$-fixed points of $X_{\mathcal{A}}$ are all isolated, and have an one-to-one correspondence with the top dimensional cones in Δ_β. For $\sigma \in \Delta_\beta$ a top dimensional cone, let $(X_{\mathcal{A}})_\sigma$ denote the fixed point corresponding to σ. Let $\sigma, \sigma' \in \Delta_\beta$ be two top dimensional cones, we write $\sigma|\sigma'$ if they intersect along a codimension-1 face. Denote by j the unique index such that $\overline{b}_j \in \sigma$, $\overline{b}_j \notin \sigma'$; and j' such that $\overline{b}_{j'} \in \sigma'$, $\overline{b}_{j'} \notin \sigma$.

Recall the box element $(f,\tau) \in \mathrm{Box}$ is given by $f = \sum_{i=1}^m f_i \overline{b}_i$ for $0 \leq f_i \leq 1$. Note that $f_i = 0$ if $i \notin \tau$.

The following is analogous to [14, Proposition 10]:

Proposition 2.18. *Let $X_{\mathcal{A}}$ be the hypertoric Deligne-Mumford stack associated to a stacky hyperplane arrangement $\mathcal{A}$. Suppose that $\sigma, \sigma' \in \Delta_\beta$ such that $\sigma|\sigma'$, and $(f,\tau) \in Box$ such that $\tau \subset \sigma$. Then the following are equivalent:*

(1) A representable morphism $g : \mathbb{P}^1_{s_1,s_2} \to X_{\mathcal{A}}$ such that $f(0) = (X_{\mathcal{A}})_\sigma$, $f(\infty) = (X_{\mathcal{A}})_{\sigma'}$, and the restriction $g|_0 : B\mu_{s_1} \to (X_{\mathcal{A}})_\sigma$ gives f.
(2) A positive rational number c such that $\langle c \rangle = f_j$.

Proof. Since any morphism $\mathbb{P}^1_{s_1,s_2} \to X_{\mathcal{A}}$ must have image inside an irreducible component of the core, which is a toric Deligne-Mumford stack, we are reduced to the case in [14, Proposition 10]. □

Remark 2.19. As in [14, Remark 11], the morphism in the above Proposition actually determines both s_2 and the box element f' given by the restriction $g|_\infty : B\mu_{s_2} \to (X_{\mathcal{A}})_{\sigma'}$. Moreover they satisfy the following relation

$$f + \lfloor c \rfloor b_j + q' b_{j'} + f' \equiv 0 \bmod \bigoplus_{i \in \sigma \cap \sigma'} \mathbb{Z} b_i \tag{2.18}$$

for some $q' \in \mathbb{Z}_{\geq 0}$.

Hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ can be deformed into a hypertoric Deligne-Mumford stack X_λ corresponding to subregular parameter λ (see §3.2), so that the curve classes are all lie in closed substacks $\overline{\mathcal{M}}^S$ associated to circuits S. So any morphism $\mathbb{P}^1_{s_1,s_2} \to X_{\mathcal{A}}$ must have image inside $\overline{\mathcal{M}}^S$, hence actually inside the weighted projective stack $\mathbb{P}^{|S|-1}_{\mathbf{w}}$. The above degree formula (2.18) can be written down in terms of the weights $\mathbf{w} = (w_1, \cdots, w_n)$.

3. Deformation of hypertoric Deligne-Mumford stacks

In this section we consider the deformation of symplectic Deligne-Mumford stacks in the hypertoric case in order to compute its Gromov-Witten invariants. For this purpose, we first recall the construction of hypertoric Deligne-Mumford stacks in symplectic category by [24], [21].

3.1. **Symplectic construction.**

In §2.1.3, the hypertoric Deligne-Mumford stack is constructed as a quotient stack $[Y/G]$, where $Y \subset T^*\mathbb{C}^m$ is a locally closed subvariety. Hypertoric Deligne-Mumford stack in symplectic category can be constructed in a similar way. Let $\mathbb{H} := T^*\mathbb{C}^m$ be the cotangent bundle of $\mathbb{C}^m$, which is a hyperkähler manifold with a canonical holomorphic symplectic form $\omega_{\mathbb{C}}$. The torus $T = (\mathbb{C}^\times)^m$ acts on $\mathbb{H}$ by diagonal on $\mathbb{C}^m$ and minus the diagonal on the fibre.

We introduce the hyperkähler T-moment map $\widetilde{\mu} = (\widetilde{\mu}_{\mathbb{R}}, \widetilde{\mu}_{\mathbb{C}})$ on $\mathbb{H}$. Let $\{t_i\}_{i=1}^m$ be a dual basis to the basis $\{\epsilon_i\}_{i=1}^m$ in $(\mathfrak{t}^m)^*$, where $\mathfrak{t}^m$ is the Lie algebra of T. Then

$$\widetilde{\mu}_{\mathbb{R}}(z,w) = \frac{1}{2}\sum_{i=1}^m (\|z_i\|^2 - \|w_i\|^2) t_i \in (\mathfrak{t}^m)^*; \tag{3.1}$$

$$\widetilde{\mu}_{\mathbb{C}}(z,w) = \sum_{i=1}^m z_i w_i t_i \in (\mathfrak{t}^m_{\mathbb{C}})^*.$$

Recall in our stacky hyperplane arrangement $\mathcal{A} = (N, \beta, \theta)$, $\mathcal{H} = \{H_1, \cdots, H_m\}$ in $M_{\mathbb{R}}$ defines a hyperplane arrangement (weighted hyperplane arrangement in [21]) in $(\mathfrak{t}^m)^* = M_{\mathbb{R}}$. Then there are exact sequences similar to the ones on §2.1.1:

$$0 \to \mathfrak{t}^{m-d} \longrightarrow \mathfrak{t}^m \xrightarrow{\beta} \mathfrak{t}^d \to 0, \tag{3.2}$$

where $\beta(\epsilon_i) = \overline{b}_i$; and

$$0 \to (\mathfrak{t}^d)^* \xrightarrow{\beta^\vee} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} (\mathfrak{t}^{m-d})^* \to 0, \tag{3.3}$$

where $\iota^*(t_i) = \lambda_i$. The T-moment map in (3.1) induces a subtorus $(\mathbb{C}^\times)^{m-d}$-moment map by:

$$\mu_{\mathbb{R}}(z,w) = \iota^* \left(\frac{1}{2} \sum_{i=1}^m (\|z_i\|^2 - \|w_i\|^2) t_i \in (\mathfrak{t}^m)^* \right); \tag{3.4}$$

$$\mu_{\mathbb{C}}(z,w) = \iota^* \left(\sum_{i=1}^m z_i w_i t_i \in (\mathfrak{t}^m_{\mathbb{C}})^* \right).$$

The hyperkähler moment map is surjective onto $(\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbb{C}})^*$. Let $(\theta, \lambda) \in (\mathfrak{t}^{m-d})^* \oplus (\mathfrak{t}^{m-d}_{\mathbb{C}})^*$ be a regular value. The hypertoric Deligne-Mumford stack (actually in this case hypertoric orbifold) is then defined by

$$\mathcal{M}(\mathcal{H}) = \mathbb{H}////_{(\theta,\lambda)} T^{m-d}$$

which is the hyperkähler reduction of $\mathbb{H}$ by T^{m-d}. We think of this as a GIT quotient stack

$$\mathcal{M}_\lambda(\mathcal{H}) = [\mu_{\mathbb{C}}^{-1}(\lambda)/_\theta T^{m-d}]$$

by the stability parameter θ.

Remark 3.1. The algebraic version of hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ defined in §2.1.3 is the case that $\lambda = 0$.

In general, if N has torsion $N_{\mathrm{tor}} = \mu$, which is a finite abelian group, $X_{\mathcal{A}}$ is a μ-gerbe over the orbifold $\mathcal{M}_\lambda(\mathcal{H})$.

3.2. **Deformations.** We adopt the argument by Konno [33] as reviewed in [40, Section 4.1].

According to [33] and [40, Section 4.1], we call a parameter $\lambda \in (\mathfrak{t}^{m-d}_{\mathbb{C}})^*$ *sub-regular* if λ lies on a unique root hyperplane

$$K_S = \mathrm{span}(\iota^* e_i^\vee : i \notin S)$$

for $S \subset \mathcal{A} = \{1, \cdots, m\}$. It is easy to check that S is a circuit.

Proposition 3.2 ([33], Theorem 5.10)**.** *Let $(z,w) \in \mu_{\mathbb{C}}^{-1}(\lambda)$. Then (z,w) is θ-stable if and only if either if the following conditions hold:*

(1) $z_i \neq 0$ for some $i \in S^+$;
(2) $w_i \neq 0$ for some $i \in S^-$.

For a sub-regular λ, the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ can be defined to be $\mathcal{M}_\lambda(\mathcal{H})$ with good properties:

Proposition 3.3. *$\mathcal{M}_\lambda(\mathcal{H})$ contains a codimension $|S|-1$ substack $\overline{\mathcal{M}}^S$, which is a weighted projective $\mathbb{P}^{|S|-1}_{\mathbf{w}}$-bundle over an affine hypertoric stack $\overline{\mathcal{M}}^S_0$ for a circuit $S \subset \mathcal{A}$. All positive dimensional projective substacks in $\mathcal{M}_\lambda(\mathcal{H})$ are contained in $\overline{\mathcal{M}}^S$.*

Proof. Let $S \subset \mathcal{A}$ be a circuit. Define the following space:

$$\mathcal{P}^S := \{w_i = 0 : i \in S^+; z_i = 0 : i \in S^-\} \subset \mathbb{H}.$$

The substack $\mathcal{M}^S$ is the GIT quotient stack

$$\overline{\mathcal{M}}^S = [(\mathcal{P}^S \cap \mu_{\mathbb{C}}^{-1}(\lambda))/_\theta T^{m-d}].$$

Now let

$$p : \mathfrak{t}^m_{\mathbb{C}} \to \mathbb{C}^{m-|S|}$$

denote the projection to the last $m-|S|$ coordinates. Then the following are true: Recall that

$$\beta_S = \sum_{i \in S^+} w_i e_i - \sum_{j \in S^-} w_j e_j.$$

Then we have:

(1) $\ker_{\mathfrak{t}^{m-d}_{\mathbb{C}}} = \mathbb{C} \cdot \beta_S$;
(2) $(p(\mathfrak{t}^{m-d}_{\mathbb{C}}))^*$ is canonically identified with $K_S \subset (\mathfrak{t}^{m-d}_{\mathbb{C}})^*$;
(3) $\lambda \in (p(\mathfrak{t}^{m-d}_{\mathbb{C}}))^*$.

Consider

$$p|_{T=(\mathbb{C}^\times)^m} : T \to (\mathbb{C}^\times)^{m-|S|}$$

and

$$\mathbb{H} \to T^*\mathbb{C}^{m-|S|}$$

by $(z_i, w_i) \mapsto (z_i, w_i)_{i \notin S}$. We have $p|_T(T^{m-d})$ acts on $T \to (\mathbb{C}^\times)^{m-|S|}$ with moment map $\mu_{\mathbb{C}}|_{m-|S|}$. So the hypertoric Deligne-Mumford stack is

$$\overline{\mathcal{M}}^S_0 = [\mu_{\mathbb{C}}|^{-1}_{m-|S|}(\lambda)/_\theta T^{m-d}].$$

Since $\lambda \in K_S$, θ is zero and the stack is affine.

By Theorem 3.2, if $(z,w) \in (\mathcal{P}^S \cap \mu_{\mathbb{C}}^{-1}(\lambda))$, then $p(z,w) \in \mu_{\mathbb{C}}|^{-1}_{m-|S|}(\lambda)$. Then we have a morphism of quotient stacks

$$\eta_S : \overline{\mathcal{M}}^S \to \overline{\mathcal{M}}^S_0.$$

Let $\mathbb{C}^{|S|} = \{z_i : i \in S^+; w_i : i \in S^-\}$. The fibre of η_S is isomorphic to the GIT quotient stack $[\mathbb{C}^{|S|}/_\theta \mathbb{C}^\times]$, where $\mathbb{C}^\times = \ker : T^{m-d} \to p(T^{m-d})$. Note that the action of $\mathbb{C}^\times$ on $\mathbb{C}^{|S|}$ is given by the weights w_i. So it is the weighted projective stack $\mathbb{P}^{|S|-1}_{\mathbf{w}}$. Any positive projective substacks of $\mathcal{M}_\lambda(\mathcal{H})$ corresponds to T^{m-d} orbits in $\mathbb{H}$ whose closure interests with the unstable locus, and they are contained in $\mathcal{P}^S$. Hence the positive dimensional substacks must be contained in $\overline{\mathcal{M}}^S$. □

4. Steinberg correspondence

In this section we discuss the stacky version of the Steinberg correspondence.

4.1. **General set-up.** From our construction of hypertoric Deligne-Mumford stack in §2.1.3, one can take the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ as a symplectic resolution of symplectic singularities. We give such a construction.

Recall that $X_{\mathcal{A}}$ is a quotient stack $[Y/G]$, where $Y \subset \mathbb{C}^{2m} \setminus V(\mathcal{I}_\theta)$ is a closed subvariety of $\mathbb{H} := T^*\mathbb{C}^m$, determined by the ideal (2.9). Let $\overline{Y} \subset \mathbb{H}$ be the subvariety determined by the ideal $I_{\beta^\vee}$ in (2.9). From the GIT point of view, $V(\mathcal{I}_\theta)$ is the unstable locus so that deleting them we have a good GIT quotient. Consider the quotient stack

$$X^0_{\mathcal{A}} = [\overline{Y}/G].$$

It is in general a singular affine stack, and not Deligne-Mumford. Let $\overline{X}^0_{\mathcal{A}}$ be the good moduli space of the stack $X^0_{\mathcal{A}}$ in the sense of [2]. The inclusion $Y \hookrightarrow \overline{Y}$ induces a projective morphism of quotient stacks, which is birational. Then we take the natural morphism

$$X_{\mathcal{A}} \to \overline{X}^0_{\mathcal{A}}$$

as a "symplectic resolution". To simplify notation we denote by $X_0 := X^0_{\mathcal{A}}, \overline{X}_0 = \overline{X}^0_{\mathcal{A}}$. The Steinberg variety is constructed by the fibre product

$$X_{\mathcal{A}} \times_{\overline{X}_0} X_{\mathcal{A}}. \tag{4.1}$$

Each component Z in (4.1) is of dimension $\leq \dim(X_{\mathcal{A}})$ (see [7], [19]). The components of dimension $\dim(X_{\mathcal{A}})$ are Lagrangian inside $X_{\mathcal{A}} \times X_{\mathcal{A}}$. Let $\mathbf{Z} \subset X_{\mathcal{A}} \times X_{\mathcal{A}}$ be all the components of Lagrangian cycles. The Steinberg correspondence

$$L : H^*_{T\times\mathbb{C}^\times}(X_{\mathcal{A}}) \to H^*_{T\times\mathbb{C}^\times}(X_{\mathcal{A}}) \tag{4.2}$$

for the $\mathbb{T}$-equivariant cohomology of $X_{\mathcal{A}}$ is given by:

$$L(\alpha) = p_{2\star}(\mathbf{Z} \cup p_1^\star \alpha),$$

where $p_i : X_{\mathcal{A}} \times X_{\mathcal{A}} \to X_{\mathcal{A}}$ is the projection for $i = 1, 2$.

In the smooth variety case, the Lagrangian components acts by correspondence on the Borel-Moore homology of $X_{\mathcal{A}}$. We hope that this is still true for Chen-Ruan cohomology. In this section we prove the case of cotangent bundle of weighted projective stacks, which we use for the calculation of quantum product. Let

$$[I\mathbf{Z}] \in H_*(I(X_{\mathcal{A}} \times X_{\mathcal{A}}))$$

be a cycle. Then by Poincare duality it defines a cohomology class in $H^*_{\mathbb{T}}(I(X_{\mathcal{A}} \times X_{\mathcal{A}}))$. The Steinberg correspondence in Chen-Ruan cohomology is:

$$\mathbf{L} : H^*_{\mathrm{CR}, T\times\mathbb{C}^\times}(X_{\mathcal{A}}) \to H^*_{\mathrm{CR}, T\times\mathbb{C}^\times}(X_{\mathcal{A}}) \tag{4.3}$$

for equivariant cohomology of $X_{\mathcal{A}}$ is given by:

$$\mathbf{L}(\alpha) = \mathrm{inv}^\star I p_{2\star}([I\mathbf{Z}] \cup I p_1^\star \alpha),$$

where $Ip_i : I(X_{\mathcal{A}} \times X_{\mathcal{A}}) \to IX_{\mathcal{A}}$ is the projection of inertia stacks for $i = 1, 2$.

4.2. **The case of cotangent bundle $T^*\mathbb{P}^n_{\mathbf{w}}$.** Let $X_{\mathcal{A}} = T^*\mathbb{P}^n_{\mathbf{w}}$ be the cotangent bundle of weighted projective stack. Then the Lagrangian components of (4.1) consists of two components:

$$X_{\mathcal{A}}; \quad \mathbb{P}^n_{\mathbf{w}} \times \mathbb{P}^n_{\mathbf{w}}.$$

It is easy to see that they are Lagrangian in $X_{\mathcal{A}} \times X_{\mathcal{A}}$. Since we are dealing with Chen-Ruan orbifold cohomology of $X_{\mathcal{A}}$, we actually need to find a correspondence in

orbifold version. For this purpose we consider the inertia stack $IX_{\mathcal{A}}$ and $I(\mathbb{P}^n_{\mathbf{w}} \times \mathbb{P}^n_{\mathbf{w}})$ inside the inertia stack $I(X_{\mathcal{A}} \times X_{\mathcal{A}})$.

Recall the the twisted sectors of weighted projective stack $\mathbb{P}^n_{\mathbf{w}}$ corresponds to finite set $F = \{\frac{d}{w_i} | 0 \leq d < w_i\}$. The twisted sectors of the product $\mathbf{Z} := \mathbb{P}^n_{\mathbf{w}} \times \mathbb{P}^n_{\mathbf{w}}$ are also indexed by pairs (α, β), where $\alpha = e^{\frac{d}{w_i}}$ and $\beta = e^{\frac{d'}{w_j}}$. The pair (α, α^{-1}) means that they are inverse in local group, i.e. if $\alpha = e^{\frac{d}{w_i}}$, then $\alpha^{-1} = e^{\frac{w_i - d}{w_i}}$. Set

$$\Gamma = \bigcup_{(\alpha,\alpha')} \mathbf{Z}_{(\alpha,\alpha')}. \tag{4.4}$$

Let $[\mathbf{w} : d] := \{w_i | d \text{ divides } w_i\}$. Then $\mathbb{P}([\mathbf{w} : d])$ is a sub-weighted projective stack of $\mathbb{P}^n_{\mathbf{w}}$. and $\mathbf{Z}_{(\alpha,\alpha')} = \mathbb{P}([\mathbf{w} : d]) \times \mathbb{P}([\mathbf{w} : d'])$.

Proposition 4.1. *All of the twisted sectors of $X_{\mathcal{A}}$ in $IX_{\mathcal{A}}$ have orbifold degree* $\dim(X_{\mathcal{A}})$ *and $IX_{\mathcal{A}}$ gives an endomorphism for the Chen-Ruan orbifold cohomology of $X_{\mathcal{A}}$.*

All of the components of $I(\mathbb{P}^n_{\mathbf{w}} \times \mathbb{P}^n_{\mathbf{w}})$ has orbifold degree $\dim(X_{\mathcal{A}})$. *Furthermore, the cycle Γ in the inertia stack defines an endomorphism for $\mathbb{T}$-equivariant Chen-Ruan orbifold cohomology of $X_{\mathcal{A}}$.*

Proof. We first do the case $IX_{\mathcal{A}}$. Note that all the components of $IX_{\mathcal{A}}$ are cotangent bundle of $T^*\mathbb{P}([\mathbf{w} : d])$, where d is a positive integer and $v = \frac{d}{w_i}$ is the local group element determining the twisted sector. The age of this element

$$\operatorname{age}(v) = \overline{A}_d,$$

where A_d denote the number of $\{w_i\}$'s such that $d | w_i$, and $\overline{A}_d$ is $n + 1 - A_d$. Note that $T^*\mathbb{P}([\mathbf{w} : d]) \subset (X_{\mathcal{A}} \times X_{\mathcal{A}})_{(v,v)} = T^*\mathbb{P}([\mathbf{w} : d]) \times T^*\mathbb{P}([\mathbf{w} : d])$ as a diagonal. Then the orbifold degree of $(X_{\mathcal{A}})_v$ inside $I(X_{\mathcal{A}} \times X_{\mathcal{A}})$ is:

$$2(A_d - 1) + \overline{A}_d + \overline{A}_d = n + 1 + n + 1 - 2 = 2n = \dim(X_{\mathcal{A}}).$$

Note that such a component in the correspondence:

$$\begin{tikzcd}
 & X_{\mathcal{A}} \subset X_{\mathcal{A}} \times X_{\mathcal{A}} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
X_{\mathcal{A}} & & X_{\mathcal{A}}
\end{tikzcd}$$

always gives identity. So $X_{\mathcal{A}}$ gives an endomorphism for the $\mathbb{T}$-equivariant Chen-Ruan cohomology.

For the other component $\mathbf{Z}$,

$$\begin{tikzcd}
 & \mathbf{Z} \subset X_{\mathcal{A}} \times X_{\mathcal{A}} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
X_{\mathcal{A}} & & X_{\mathcal{A}}.
\end{tikzcd} \tag{4.5}$$

We prove that each component in $I\mathbf{Z}$ has orbifold degree of dimension of $X_{\mathcal{A}}$. A twisted sector of $\mathbf{Z}$ corresponds to a pair (v_1, v_2) where $v_1 = \frac{d_1}{w_i}$, $v_2 = \frac{d_2}{w_j}$. And

$$\mathbf{Z}_{(v_1,v_2)} = \mathbb{P}([\mathbf{w} : d_1]) \times \mathbb{P}([\mathbf{w} : d_2]) \subset (X_{\mathcal{A}})_{v_1} \times (X_{\mathcal{A}})_{v_2} = T^*\mathbb{P}([\mathbf{w} : d_1]) \times T^*\mathbb{P}([\mathbf{w} : d_2]).$$

So the orbifold degree of $\mathbf{Z}_{(v_1,v_2)}$ is:

$$\begin{aligned} A_{d_1}-1+A_{d_2}-1+\mathrm{age}(v_1)+\mathrm{age}(v_2) &= A_{d_1}-1+A_{d_2}-1+\overline{A}_{d_1}+\overline{A}_{d_2} \\ &= n+1-1+n+1-1 \\ &= 2n. \end{aligned}$$

We may call all the components of $I\mathbf{Z}$ Lagrangian cycle in the correspondence of Chen-Ruan cohomology. Now each component in Γ corresponds to

$$\begin{tikzcd} & \mathbf{Z}_{(\alpha,\alpha')} \subset (X_{\mathcal{A}})_\alpha \times (X_{\mathcal{A}})_{\alpha'} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ (X_{\mathcal{A}})_\alpha & & (X_{\mathcal{A}})_{\alpha'}. \end{tikzcd} \tag{4.6}$$

The above diagram actually corresponds to lower dimensional cotangent bundle of weighted projective stacks, hence similar argument as in [7], [19] proves that $\mathbf{Z}_{(\alpha,\alpha')}$ gives an endomorphism on $\mathbb{T}$-equivariant cohomology. Chen-Ruan cohomology is a sum of all of the twisted sectors like this, hence Γ gives an endomorphism on $\mathbb{T}$-equivariant Chen-Ruan cohomology. $\square$

Example 4.2. We compute an example $X_{\mathcal{A}} = T^*\mathbb{P}(1,2)$. In this example the components of $I\mathbf{Z} = I(\mathbb{P}(1,2)\times\mathbb{P}(1,2))$ are given by:

$$\begin{aligned} &\mathbf{Z}_{(0,0)} = \mathbf{Z}; \quad \mathbf{Z}_{(0,1/2)} = \mathbb{P}(1,2)\times B\mu_2; \\ &\mathbf{Z}_{(1/2,0)} = B\mu_2\times\mathbb{P}(1,2); \quad \mathbf{Z}_{(1/2,1/2)} = B\mu_2\times B\mu_2. \end{aligned}$$

It is not hard that we can check that Γ gives an endomorphism for the Chen-Ruan cohomology of $X_{\mathcal{A}}$.

We compute that using $I\mathbf{Z}$. Consider

$$\begin{tikzcd} & \mathbf{Z}_{(\alpha,\beta)} \subset (X_{\mathcal{A}})_\alpha \times (X_{\mathcal{A}})_{\beta} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ (X_{\mathcal{A}})_\alpha & & (X_{\mathcal{A}})_{\beta}. \end{tikzcd}$$

We apply the above diagram to every component in $I\mathbf{Z}$. The equivariant Chen-Ruan cohomology of $X_{\mathcal{A}}$ is:

$$H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}}) = \frac{\mathbb{Q}[u_1,u_2,\hbar,\mathbb{1}_{\frac{1}{2}}]}{\{u_1\cdot u_2, \mathbb{1}_{\frac{1}{2}}u_1, \mathbb{1}_{\frac{1}{2}}u_2, \mathbb{1}_{\frac{1}{2}}\cdot\mathbb{1}_{\frac{1}{2}} = u_1^2\}}.$$

We calculate:

$$\begin{aligned} \mathbf{L}(u_1) &= \frac{1}{2}(\hbar-u_1-u_2)+\frac{1}{2}\mathbb{1}_{\frac{1}{2}}; \\ \mathbf{L}(u_2) &= (\hbar-u_1-u_2)+\frac{1}{2}\mathbb{1}_{\frac{1}{2}}; \\ \mathbf{L}(\mathbb{1}_{\frac{1}{2}}) &= \frac{1}{2}(\hbar-u_1-u_2)+\frac{1}{2}\mathbb{1}_{\frac{1}{2}}. \end{aligned}$$

We calculate $\mathbf{L}^{-1}$, where

$$\mathbf{L}^{-1} := (Ip_1)_\star([I\mathbf{Z}]\cup Ip_2^\star(-)).$$

To do that we use localization. There are two $T \times \mathbb{C}^\times$ fixed points on $X_\mathcal{A}$, which is $P_1 = [1, 0] \in \mathbb{P}(1, 2), P_2 = [0, 1] \in \mathbb{P}(1, 2)$. We calculate:

$$e(N_{P_1} X_\mathcal{A}) = \lambda_1(\hbar - \lambda_1 - \lambda_2); \quad e(N_{P_2} X_\mathcal{A}) = \lambda_2(\hbar - \lambda_1 - \lambda_2).$$

We first calculate:

$$\begin{aligned}\int_{X_\mathcal{A}} (\hbar - u_1 - u_2)^2 &= \frac{1}{2}\left(\frac{(\hbar - \lambda_1 - \lambda_2)^2}{\lambda_1(\hbar - \lambda_1 - \lambda_2)} + \frac{(\hbar - \lambda_1 - \lambda_2)^2}{\lambda_2(\hbar - \lambda_1 - \lambda_2)}\right)\\ &= \frac{1}{2}\left(\frac{(\hbar - \lambda_1 - \lambda_2)}{\lambda_1} + \frac{(\hbar - \lambda_1 - \lambda_2)}{\lambda_2}\right)\\ &= \frac{1}{2}\left(\frac{(\hbar - \lambda_2)}{\lambda_1} + \frac{(\hbar - \lambda_1)}{\lambda_2} - 2\right).\end{aligned}$$

Hence we have:

$$\begin{aligned}\mathbf{L}^{-1}(\hbar - u_1 - u_2) = \frac{1}{2}\left(\frac{(\hbar - \lambda_2)}{\lambda_1} + \frac{(\hbar - \lambda_1)}{\lambda_2} - 2\right)(\hbar - u_1 - u_2)\\ + \frac{1}{2}\left(\frac{(\hbar - \lambda_2)}{\lambda_1} + \frac{(\hbar - \lambda_1)}{\lambda_2} - 2\right)\mathbb{1}_{\frac{1}{2}};\\ \mathbf{L}^{-1}(\mathbb{1}_{\frac{1}{2}}) = \frac{1}{2}(\hbar - u_1 - u_2) + \frac{1}{2}\mathbb{1}_{\frac{1}{2}}.\end{aligned}$$

So

$$\begin{aligned}\mathbf{L}^{-1}(\mathbf{L}(u_1)) = \frac{1}{4}\left(\frac{(\hbar - \lambda_2)}{\lambda_1} + \frac{(\hbar - \lambda_1)}{\lambda_2} - 2\right)(\hbar - u_1 - u_2)\\ + \frac{1}{4}\left(\frac{(\hbar - \lambda_2)}{\lambda_1} + \frac{(\hbar - \lambda_1)}{\lambda_2} - 2\right)\mathbb{1}_{\frac{1}{2}} + \frac{1}{4}(\hbar - u_1 - u_2) + \frac{1}{4}\mathbb{1}_{\frac{1}{2}}.\end{aligned}$$

It is seen that $\mathbf{L}^{-1}\mathbf{L}$ is not an identity. But $\mathbf{L} : H^*_{\mathrm{CR},\mathbb{T}}(X_\mathcal{A}) \to H^*_{\mathrm{CR},\mathbb{T}}(X_\mathcal{A})$ is injective.

4.3. **The case of deformation $\mathcal{M}_\lambda(\mathcal{H})$.** Let $\mathcal{M}_\lambda(\mathcal{H})$ be the deformation of $X_\mathcal{A}$ by sub-regular parameter λ. We fix a notation that $\mathfrak{M}_\lambda := \mathcal{M}_\lambda(\mathcal{H})$. Then there is a contraction map from $\mathfrak{M}_\lambda \to \mathfrak{M}^0_\lambda$ such that

$$\mathfrak{M}_\lambda \times_{\mathfrak{M}^0_\lambda} \mathfrak{M}_\lambda \subset \mathfrak{M}_\lambda \times \mathfrak{M}_\lambda$$

contains components whose dimension are the same as $\dim(\mathcal{M}_\lambda(\mathcal{H}))$.

Let $(v, \sigma) \in \mathrm{Box}$ be an element in the box. Then $(X_\mathcal{A})_{(v,\sigma)} := X_{\mathcal{A}/\sigma}$ is again a hypertoric Deligne-Mumford stack, associated to the quotient stacky hyperplane arrangement $\mathcal{A}/\sigma = (N(\sigma), \beta(\sigma), \theta(\sigma))$. Then the twisted sector $(\mathfrak{M}_\lambda)_{(v,\sigma)}$ of the deformation $\mathcal{M}_\lambda(\mathcal{H})$, associated to (v, σ), is also a hypertoric Delinge-Mumford stack, which is the deformation of $X_{\mathcal{A}/\sigma}$. Let $\mathbf{Z} := \mathfrak{M}_\lambda \times_{\mathfrak{M}^0_\lambda} \mathfrak{M}_\lambda$. Then similar to (4.3), the inertia stack $I\mathbf{Z}$ gives the Steinberg correspondence:

$$\mathbf{L} := L_{I\mathbf{Z}} : H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{M}_\lambda) \to H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{M}_\lambda). \tag{4.7}$$

5. Quantum product by divisors

In this section we calculate the small equivariant quantum product by divisors for hypertoric Deligne-Mumford stacks, which proves Theorem 1.1. The quantum product by divisors is reduced to the calculation of 2-point Gromov-Witten invariants. The computation of quantum product by divisors for smooth hypertoric varieties in [40] uses a result stated as [40, Proposition 2.2], which was proven in [7] in general setting

of symplectic resolutions. We prove a similar result for hypertoric Deligne-Mumford stacks. The arguments here should work for more general *stacky symplectic resolutions*.

5.1. **Reduced virtual fundamental cycle on the deformation.** Let $X_{\mathcal{A}}$ be a hypertoric Deligne-Mumford stack associated with a stacky hyperplane arrangement $\mathcal{A}$. As in §3.2, deformation of $X_{\mathcal{A}}$ is obtained by varying the level of moment map parameter $\lambda \in (\mathfrak{t}_{\mathbb{C}}^{m-d})^*$. Our algebraic construction $X_{\mathcal{A}}$ of hypertoric Deligne-Mumford stack corresponds to $\lambda = 0$. Each circuit $S \subset \mathcal{A}$ gives a *root hyperplane* $K_S = \mathrm{span}(\iota^* e_i^\vee : i \notin S) \subset (\mathfrak{t}_{\mathbb{C}}^{m-d})^*$. By divisor equation, we are interested in the equivariant virtual fundamental cycle $[\overline{\mathcal{M}}_{0,2}(X_{\mathcal{A}}, d)]^{\mathrm{virt}}$. By (2.17) we have:

$$[\overline{\mathcal{M}}_{0,2}(X_{\mathcal{A}}, d)]^{\mathrm{virt}} = \hbar \cdot [\overline{\mathcal{M}}_{0,2}(X_{\mathcal{A}}, d)]^{\mathrm{red}}.$$

Let $\phi_0 : \mathbb{A}^1 \to (\mathfrak{t}_{\mathbb{C}}^{m-d})^*$ be a generic linear subspace such that it intersects every hyperplane transversely exactly once at the origin. The deformation of $X_{\mathcal{A}}$ over $(\mathfrak{t}_{\mathbb{C}}^{m-d})^*$ in §3.2 restricts to give a smooth map $\mathcal{V}_0 \to \mathbb{A}^1$ whose fibre over the origin is $X_{\mathcal{A}}$. The reduced virtual fundamental cycle in §2.3.2 has another explanation which was developed in [7, Proposition 4.1]. We stated it as:

$$[\overline{\mathcal{M}}_{0,2}(X_{\mathcal{A}}, d)]^{\mathrm{red}} = [\overline{\mathcal{M}}_{0,2}(\mathcal{V}_0, d)]^{\mathrm{virt}}.$$

We deform $\mathcal{V}_0 \to \mathbb{A}^1$ again. Put $D := \mathbb{A}^1$. Choose a family of maps parametrized by $t \in D$, $\phi_t : \mathbb{A}^1 \to (\mathfrak{t}_{\mathbb{C}}^{m-d})^*$, such that for $t = 0$ we have the ϕ_0 above, and for $t \neq 0$ sufficiently close to 0, the image of ϕ_t intersects each hyperplane transversely at distinct points. Write $\mathcal{V}_t \to \mathbb{A}^1$ for the family obtained by restriction to ϕ_t. The deformation invariance implies that

$$[\overline{\mathcal{M}}_{0,2}(\mathcal{V}_0, d)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,2}(\mathcal{V}_t, d)]^{\mathrm{virt}}.$$

By construction, compact curves can only be in the fibres of $\mathcal{V}_t \to \mathbb{A}^1$ over points of intersects with root hyperplanes. We conclude that only multiples of β_S appears in the quantum corrections. To study the quantum correction from class $m\beta_S$, we study $\mathcal{M}_\lambda(\mathcal{H})$ where λ is subregular (i.e., lying on K_S and only K_S). Proposition 3.3 implies that compact curves lie in the fibres of $\mathcal{M}^S \to \mathcal{M}_0^S$. Since $\mathcal{M}^S$ is symplectic and the fibres are isotropic, its normal bundle is identified with the cotangent bundle. So this reduces the calculation of equivariant Gromov-Witten invariants to the case of cotangent bundle $T^*\mathbb{P}^n_{\mathbf{w}}$ of $\mathbb{P}^n_{\mathbf{w}}$.

5.2. **Broken and unbroken twisted maps.** We list some properties of broken and unbroken maps as in [41, Section 3.8.2], see also [39, Section 7.3]. Consider the moduli stack $\overline{\mathcal{M}}_{0,k}(X_{\mathcal{A}}, d)$ for $d > 0$ with the torus $\mathbb{T} := T \times \mathbb{C}^\times$-action. In each $\mathbb{T}$-fixed twisted stable map $f : \mathcal{C} \to X_{\mathcal{A}}$ in $\overline{\mathcal{M}}_{0,k}(X_{\mathcal{A}}, d)$, the twisted curve $\mathcal{C}$ is a chain of twisted rational curves

$$\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$$

with k markings. Let $p \in \mathcal{C}$ be a marking, the $T \times \mathbb{C}^\times$ weight ω_p at p is defined by the $T \times \mathbb{C}^\times$-representation of the tangent space to $\mathcal{C}$ at p. Similarly if $\mathcal{P} \subset \mathcal{C}$ is a component incident to a twisted node s. A $T \times \mathbb{C}^\times$ weight $\omega_{\mathcal{P},s}$ at s is defined by the $T \times \mathbb{C}^\times$-representation of the tangent space to $\mathcal{P}$ at s.

If at a node s, the $\mathbb{T}$ weights of the two branches are opposite and nonzero then we say that f is an *unbroken chain*. Moreover we say that f is an *unbroken twisted map* if it satisfies one of the following three conditions:

(1) f comes from a twisted map $f:\mathcal{C}\to X^{\mathbb{T}}_{\mathcal{A}}$,
(2) f is an unbroken chain,
(3) the curve $\mathcal{C}$ is a chain of twisted rational curves

$$\mathcal{C}_0\cup\mathcal{C}_1\cup\cdots\cup\mathcal{C}_k$$

such that $\mathcal{C}_0$ is contracted by f, the marked points lie in $\mathcal{C}_0$ and the remaining of $\mathcal{C}$ forms a unbroken chain.

The $\mathbb{T}$-fixed twisted stable maps which do not satisfy the above conditions are called *broken twisted maps.*

5.3. **Calculation on $T^*\mathbb{P}^n_{\mathbf{w}}$.** The cotangent bundle $X:=T^*\mathbb{P}^n_{\mathbf{w}}$ is a hypertoric Deligne-Mumford stack. Recall that in §4 the components of $I\mathbf{Z}\subset IX\times IX$ define the Steinberg correspondence (4.3) for equivariant Chen-Ruan cohomology $H^*_{\mathrm{CR},\mathbb{T}}(X)$. If $\{\gamma_i\}$ is a basic for the Chen-Ruan cohomology $H^*_{\mathrm{CR},\mathbb{T}}(X)$, we let $\{\gamma^i\}$ the dual basis under orbifold Poincaré pairing. Let D be a divisor class in $H^2_{\mathbb{T}}(X,\mathbb{Q})$. In this case all curves lie in the weighted projective stack, hence $\mathrm{NE}(X)_{\mathbb{Z}}=\mathbb{Z}_{\geq 0}$. Let $\ell\subset \mathrm{NE}(X)_{\mathbb{Z}}$ be the primitive line. Recall that the quantum product is defined by:

$$(D\star\gamma_i,\gamma_j)=\sum_{d\ell\geq 0}Q^{d\ell}\langle D,\gamma_i,\gamma_j\rangle^X_{0,3,d}=\sum_{d\ell\geq 0}(\int_{d\ell}D)Q^{d\ell}\langle\gamma_i,\gamma_j\rangle^X_{0,2,d}$$

Let

$$\mathbf{ev}:=\mathrm{ev}_1\times\mathrm{ev}_2:\overline{\mathcal{M}}_{0,2}(X,d)\to IX\times IX$$

be the evaluation map. Then using the relationship:

$$[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{virt}}=\hbar\cdot[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{red}}$$

for $d\neq 0$, we can write the quantum product by D as:

(5.1)

$$\begin{aligned}D\star\gamma_i&=D\cup_{\mathrm{CR}}\gamma_i+\sum_{d\ell>0}\int_{d\ell}D\cdot Q^{d\ell}\langle\gamma_i,\gamma_j\rangle^X_{0,2,d}\gamma^j\\&=D\cup_{\mathrm{CR}}\gamma_i+\sum_{d\ell>0}\int_{d\ell}D\cdot Q^{d\ell}\cdot\hbar\cdot\mathrm{inv}^*\cdot Ip_{2\star}(Ip_1^\star(\gamma_i)\cap\mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{red}}).\end{aligned}$$

We process (5.1) by localization of the $\mathbb{T}$-action on the moduli stack.

Recall that in §4.2, the twisted sectors of the weighted projective stack $\mathbb{P}^n_{\mathbf{w}}$ are indexed by the set

$$F:=\{\frac{a}{w_i}|0\leq a<w_i,0\leq i\leq n\},$$

and the twisted sector corresponding to $f=\frac{a}{w_i}$ is the sub-weighted projective stack $\mathbb{P}([\mathbf{w}:d])$, where $d=|e^{2\pi i f}|$ is the order of $e^{2\pi i f}$. The components of the inertia stack IX of the cotangent bundle $X=T^*\mathbb{P}^n_{\mathbf{w}}$ are also indexed by the set F, and the twisted sector X_f corresponding to $f=\frac{a}{w_i}$ is the cotangent bundle $T^*\mathbb{P}([\mathbf{w}:d])$ of the sub-weighted projective stack $\mathbb{P}([\mathbf{w}:d])$.

First we have the following fact for the pushforward of virtual fundamental cycle:

Lemma 5.1.

$$\mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X,d)]^{red}\in H^{2n}_{CR,\mathbb{T}}(IX\times IX),$$

where $2n=\dim(X)$*.*

Proof. Let $\overline{\mathcal{M}}_{0,2}(X,d)^{f_1,f_2}$ be the component of the moduli stack indexed by the elements $f_1, f_2 \in F$ such that the evaluation maps $\mathrm{ev}_1, \mathrm{ev}_2$ have images in X_{f_1} and X_{f_2}, respectively. The cycle $[\overline{\mathcal{M}}_{0,2}(X,d)^{f_1,f_2}]^{\mathrm{red}}$ has dimension

$$\begin{aligned}\dim([\overline{\mathcal{M}}_{0,2}(X,d)^{f_1,f_2}]^{\mathrm{red}}) &= \dim(X) - \mathrm{age}(X_{f_1}) - \mathrm{age}(X_{f_2}) \\ &= n - \mathrm{age}(X_{f_1}) + n - \mathrm{age}(X_{f_2}) \\ &= \dim(\mathbb{P}([\mathbf{w}:d_1])) + \dim(\mathbb{P}([\mathbf{w}:d_2])),\end{aligned}$$

where d_1, d_2 are the orders of $e^{2\pi i f_1}, e^{2\pi i f_2}$. Note that

$$\mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{red}} \in H_{\mathbb{T}}^{2n-\mathrm{age}(X_{f_1})-\mathrm{age}(X_{f_2})}(X_{f_1} \times X_{f_2}).$$

Hence if f_1, f_2 belong to the same local group and have the same order, then $X_{f_1} \cong X_{f_2}$ and $\mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X,d)^{f_1,f_2}]^{\mathrm{red}}$ supports on the diagonal $X_{f_1} \times X_{f_2}$ or $\mathbb{P}([\mathbf{w}:d_1])) \times \mathbb{P}([\mathbf{w}:d_2])$. If f_1, f_2 do not have the same orders, then $\mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X,d)^{f_1,f_2}]^{\mathrm{red}}$ supports on $\mathbb{P}([\mathbf{w}:d_1])) \times \mathbb{P}([\mathbf{w}:d_2])$. $\square$

Lemma 5.1 tells us that the pushforward of reduced cycle supports on the Lagrangian cycles inside $IX \times IX$. Next we use localization of reduced virtual fundamental cycles as in [22], [39] to calculate the sign of the support. We first have the following lemma:

Lemma 5.2. *Let $\mathbb{P}^1_{s_1,s_2}$ be a $\mathbb{P}^1$-orbifold with stacky points $P_1 = [1,0] = B\mu_{s_1}$ and $P_2 = [0,1] = B\mu_{s_2}$. Assume that $\mathcal{V}$ is a vector bundle on $\mathbb{P}^1_{s_1,s_2}$. Let T be a torus acting on $\mathcal{V}$ with no zero weights. Then we have the equivariant Euler class*

$$e_T(H^0(\mathcal{V}\oplus\mathcal{V}^* - H^1(\mathcal{V}\oplus\mathcal{V}^*)) = (-1)^{\chi(\mathcal{V})+\mathrm{rk}(H^1(\mathcal{V}\oplus\mathcal{V}^*)^T}e_T(\mathcal{V}_{P_1})^{inv}\cdot e_T(\mathcal{V}_{P_2})^{inv},$$

where $(\mathcal{V}_{P_i})^{inv}$ is the invariant part of the restriction of $\mathcal{V}$ to P_i.

Proof. By a result of Martens-Thaddeus [38], the vector bundle $\mathcal{V}$ splits as a direct sum of line bundles. Then the formula comes from [36, Example 98]. $\square$

The torus $\mathbb{T}$ acts on the moduli stack $\overline{\mathcal{M}}_{0,2}(X,d)$. The following lemma is very similar to Lemma 6 in [41] (with the same proof):

Lemma 5.3. *The $\mathbb{T}$-fixed broken twisted maps contribute zero under $\mathbb{T}$-localization.*

The $\mathbb{T}$-fixed points of X are all contained in the 0-section $\mathbb{P}^n_{\mathbf{w}}$, and are the same as the fixed points set of the torus action on $\mathbb{P}^n_{\mathbf{w}}$. The torus $\mathbb{T}$-fixed one-dimensional orbits of X are contained in $\mathbb{P}^n_{\mathbf{w}}$ and gerbes of weighted projective lines. The following is a stacky version of [41, Lemma 7]:

Lemma 5.4. *There are no unbroken twisted stable maps in $\overline{\mathcal{M}}_{0,2}(X,d)^{\mathbb{T}}$ with reducible domains connecting two $\mathbb{T}$-fixed points in $\mathbb{P}^n_{\mathbf{w}}$.*

Proof. The torus fixed one-dimensional orbits of X are gerbes of weighted projective lines. Then the result follows from the fact that there are no $\mathbb{T}$-fixed unbroken twisted maps connecting the $\mathbb{T}$-fixed points of X to itself. $\square$

Look at the evaluation map $\mathbf{ev}: \overline{\mathcal{M}}_{0,2}(X,d) :\to IX \times IX$, from Lemma 5.4, the cycle $\mathbf{ev}_\star([\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{red}})$ is supported away from the diagonal of $IX \times IX$. Also the affinization $X \to \overline{X}^0$ contracts the zero section. Hence the cycle $\mathbf{ev}_\star([\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{red}})$ factors through $IX_{I\overline{X}^0}IX \subset IX \times IX$. Lemma 5.4 tells us that the only $\mathbb{T}$-fixed twisted

stable maps come from a cyclic cover map to a $\mathbb{T}$-fixed one dimensional orbit inside weighted projective stack $\mathbb{P}^n_{\mathbf{w}}$. So let $\mathbb{P}^1_{s_1,s_2} \to X$ be a $\mathbb{T}$-fixed twisted stable map to X, such that it maps to the $\mathbb{T}$-fixed one dimensional orbit inside $\mathbb{P}^n_{\mathbf{w}}$ corresponding to $f_1, f_2 \in F$. We apply the formula in Lemma 5.2 to $\mathcal{V} = f^\star T_{\mathbb{P}^n_{\mathbf{w}}}$. Note that $f^\star T_X = \mathcal{V} \oplus \mathcal{V}^*$. Let $P := \mathbb{P}^n_{\mathbf{w}}$. By localization as in [39, Chapter 11], the contribution of f to $\mathbf{ev}_\star([\overline{\mathcal{M}}_{0,2}(X, d\ell)]^{\mathrm{red}})$ is :

$$\frac{1}{\mathrm{Aut}(f)} e_{\mathbb{T}}(H^0 - H^1(\mathcal{V} \oplus \mathcal{V}^*))^{-1}$$

and the component $\mathbf{ev}_\star([\overline{\mathcal{M}}_{0,2}(X, d)^{f_1,f_2}]^{\mathrm{red}}) \in H_*(X_{f_1} \times X_{f_2})$ is:

$$\frac{1}{d}(-1)^{\mathrm{rk}(\mathcal{V}) + \int_{d\ell} c_1(T_{\mathbb{P}^n_{\mathbf{w}}})}(-1)^{-\mathrm{age}(P_{f_1})}[P_{f_1}] \times (-1)^{-\mathrm{age}(P_{f_2})}[P_{f_2}].$$

Set

$$\Gamma_{f_1,f_2} := (-1)^{-\mathrm{age}(P_{f_1})}[P_{f_1}] \times (-1)^{-\mathrm{age}(P_{f_2})}[P_{f_2}]; \quad \Gamma := \bigoplus_{f_1,f_2 \in F} \Gamma_{f_1,f_2}.$$

For fixed $f_1, f_2 \in F$, it is not hard to see from Proposition 2.18 that those degrees $d \in \mathbb{Z}_{\geq 0}$ such that there exist maps $\mathbb{P}^1_{s_1,s_2} \to P$ with stack structures specified by (f_1, f_2), are exactly of the following form[3]:

$$d = \delta \cdot \mathrm{lcm}(w_0, \cdots, w_n) + r(f_1, f_2),$$

where $\delta \in \mathbb{Z}_{\geq 0}$, $r(f_1, f_2) \in \{0, 1, \cdots, \mathrm{lcm}(w_i) - 1\}$ is a number such that $\langle \frac{r(f_1,f_2)}{w_i} \rangle = f_1$, $\langle \frac{r(f_1,f_2)}{w_j} \rangle = f_2$, and $r(f_1, f_2) \equiv 0 \mod w_k$ for $w_k \neq w_i, w_j$ (here $f_1 = \frac{a_1}{w_i}$, $f_2 = \frac{a_2}{w_j}$). From (5.1), the quantum product by divisor D is:

$$\begin{aligned}
D \star \gamma_i &= D \cup_{\mathrm{CR}} \gamma_i + \sum_{d\ell > 0} \int_{d\ell} D \cdot Q^{d\ell} \cdot \hbar \cdot \mathrm{inv}^* \cdot Ip_{2\star}(Ip_1^\star(\gamma_i) \cap \mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X, d)]^{\mathrm{red}}). \\
&= D \cup_{\mathrm{CR}} \gamma_i + \int_\ell D \cdot \sum_{d\ell > 0} Q^{d\ell} \cdot d \cdot \hbar \cdot \mathrm{inv}^* \cdot Ip_{2\star}(Ip_1^\star(\gamma_i) \cap \mathbf{ev}_\star[\overline{\mathcal{M}}_{0,2}(X, d)]^{\mathrm{red}}). \\
&= D \cup_{\mathrm{CR}} \gamma_i + \hbar \int_\ell D \cdot (-1)^n \cdot \sum_{\delta \geq 0} \sum_{(f_1,f_2) \in F^2} Q^{(\delta \cdot \mathrm{lcm}(w_i) + r(f_1,f_2))\ell} \qquad (5.2) \\
&\quad \cdot (-1)^{(\delta \cdot \mathrm{lcm}(w_i) + r(f_1,f_2)) \int_\ell c_1(T_P)} \cdot \mathrm{inv}^* \cdot Ip_{2\star}(Ip_1^\star(\gamma_i) \cap \Gamma_{f_1,f_2}). \\
&= D \cup_{\mathrm{CR}} \gamma_i + \begin{cases} \hbar \int_\ell D \cdot (-1)^n \frac{(Q^\ell (-1)^{\sum_i \frac{1}{w_i}})^{r(f_1,f_2)}}{1 - ((-1)^{\sum_i \frac{1}{w_i}} Q^\ell)^{\mathrm{lcm}(w_i)}} \Gamma_{f_1,f_2}; & r(f_1, f_2) \neq 0; \\ \hbar \int_\ell D \cdot (-1)^n \frac{(Q^\ell (-1)^{\sum_i \frac{1}{w_i}})^{\mathrm{lcm}(w_i)}}{1 - ((-1)^{\sum_i \frac{1}{w_i}} Q^\ell)^{\mathrm{lcm}(w_i)}} \Gamma_{f_1,f_2}; & r(f_1, f_2) = 0, \end{cases}
\end{aligned}$$

where in the calculation we use $\int_\ell c_1(T_P) = \sum_i \frac{1}{w_i}$, and Γ_{f_1,f_2} is the Steinberg correspondence in (4.3).

5.4. **Proof of Theorem 1.1.** Recall that in §3, the hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ can be deformed into X_λ for sub-regular parameter. Proposition 3.3 says that X_λ contains a substack $\overline{\mathcal{M}}^S \to \overline{\mathcal{M}}^S_0$ for any circuit $S \subset \mathcal{A}$, which is the weighted projective bundle over a singular base $\overline{\mathcal{M}}^S_0$. The fibre is the weighted projective stack $\mathbb{P}^{|S|-1}_{\mathbf{w}}$ with

[3] The existence of $\gamma(f_1, f_2)$ imposes constraints on the possible pairs (f_1, f_2).

fibre-wise normal bundle the cotangent bundle $T^*\mathbb{P}_{\mathbf{w}}^{|S|-1}$. Hence the curve class in the 2-point Gromov-Witten invariants must lie in $\mathbb{P}_{\mathbf{w}}^{|S|-1}$. The twisted sectors $(X_\lambda)_{f_i}$ for f_i is determined by $f_i \in F$ for $i = 1, 2$. In this case the Lagrangian cycle $\Gamma_{f_1,f_2} \subset IX_\lambda \times IX_\lambda$ is actually lying in $I\overline{\mathcal{M}}^S \times I\overline{\mathcal{M}}^S$. Then same calculation as in §5.3 concludes that we have the same as (5.2). The theorem is proved.

6. Gromov-Witten theory of hypertoric Deligne-Mumford stacks

In this section we compare the $\mathbb{T}$-equivariant Gromov-Witten theory of $X_{\mathcal{A}}$ with the $\mathbb{T}$-equivariant Gromov-Witten theory of the associated Lawrence toric Deligne-Mumford stack X_θ.

6.1. **Comparison of Gromov-Witten invariants.** Let $\mathcal{A}$ be a stacky hyperplane arrangement and $X_{\mathcal{A}}$ the corresponding hypertoric Deligne-Mumford stack. As discussed in §2.1.3, $X_{\mathcal{A}}$ is defined as a closed substack of the corresponding Lawrence toric Deligne-Mumford stack X_θ. The stacky fan $\mathbf{\Sigma}_\theta$ determines an open sub-variety $X := \mathbb{C}^{2m} \setminus V(\mathcal{I}_\theta) \subset \mathbb{C}^{2m}$ and the Lawrence toric Deligne-Mumford stack X_θ is the quotient stack $[X/G]$. Let $Y \subset X$ be the closed sub-variety determined by the ideals in (2.9). The hypertoric Deligne-Mumford stack $X_{\mathcal{A}}$ is the quotient stack $[Y/G]$.

Lemma 6.1. *The normal bundle $N := N_{X_{\mathcal{A}}/X_\theta}$ of $X_{\mathcal{A}} \subset X_\theta$ is trivial.*

Proof. This can be deduced from generalized Euler sequence in [17, Section 1.1.1]. Alternatively, N is trivial because in the equations for $X_{\mathcal{A}}$ in (2.9), z_i and w_i are sections of line bundles of X_θ that are dual to each other. □

The tori T and $\mathbb{T}$ act on both $X_{\mathcal{A}}$ and X_θ. The inclusion $\iota : X_{\mathcal{A}} \hookrightarrow X_\theta$ is equivariant with respect the the actions of these two tori.

Lemma 6.2. *The fixed loci $X_{\mathcal{A}}^T$ is the same as the fixed loci X_θ^T.*

Proof. It is clear that $X_{\mathcal{A}}^T \subset X_\theta^T$. Let T' be the one dimensional torus defined by the vector $\sum_{i=1}^m b_i \in N$. The T'-action on the Lawrence toric Deligne-Mumford stack is induced by the multiplication by non-zero complex numbers on $\mathbb{C}^{2m}$. This is the action defined in [23, Lemma 6.5]. It is shown in [23] and [17] that $X_{\mathcal{A}}^{T'} = X_\theta^{T'}$. Thus $X_\theta^T \subset X_\theta^{T'} = X_{\mathcal{A}}^{T'} = X_{\mathcal{A}}^T$, which implies that $X_\theta^T \subset X_{\mathcal{A}}^T$. □

Lemma 6.3. *The T-actions on both $X_{\mathcal{A}}$ and X_θ have identical compact one-dimensional orbits.*

Proof. The projective substack of $X_{\mathcal{A}}$ and X_θ is the core $C(X_{\mathcal{A}})$ defined in §2.1.4. Since the core $C(X_{\mathcal{A}})$ and any compact curve are contracted by the affinization map $X_\theta \to \mathrm{Spec}H^0(\mathcal{O}_{X_\theta})$, we see that the compact one-dimensional T-orbits are all contained in $C(X_{\mathcal{A}})$. □

The inclusion ι induces an inclusion of moduli stacks

$$\iota : \overline{\mathcal{M}}_{g,n}(X_{\mathcal{A}}, d) \hookrightarrow \overline{\mathcal{M}}_{g,n}(X_\theta, d),$$

which is $\mathbb{T}$-equivariant.

By [17], the pullback

$$\iota^\star : H^*_{\mathrm{CR},\mathbb{T}}(X_\theta) \to H^*_{\mathrm{CR},\mathbb{T}}(X_{\mathcal{A}})$$

is a ring isomorphism.

The following result concerns genus 0 Gromov-Witten invariants.

Proposition 6.4. *Let $\gamma_1, \cdots, \gamma_n \in H^*_{CR,\mathbb{T}}(X_\theta)$, and $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$. Then there is an equality on descendant $\mathbb{T}$-equivariant Gromov-Witten invariants*

$$\left\langle \prod_{i=1}^n \gamma_i \psi_i^{k_i} \right\rangle_{0,n,d}^{X_\theta} = \frac{\left\langle \prod_{i=1}^n \iota^\star(\gamma_i)\psi_i^{k_i} \right\rangle_{0,n,d}^{X_\mathcal{A}}}{e_\mathbb{T}(N)},$$

where $e_\mathbb{T}(N)$ is the equivariant Euler class of N. Note that the Gromov-Witten invariants take values in $H^*_\mathbb{T}(pt)$.

Proof. We compare the two invariants by $\mathbb{T}$-equivariant localization. By Lemma 6.2 and 6.3, we have $\overline{\mathcal{M}}_{g,n}(X_\mathcal{A}, d)^\mathbb{T} = \overline{\mathcal{M}}_{g,n}(X_\theta, d)^\mathbb{T}$ with identical $\mathbb{T}$-fixed obstruction theories. A simple check shows that $e_\mathbb{T}$ of the virtual normal bundle for X_θ and $X_\mathcal{A}$ differ by $e_\mathbb{T}(H^1(f^\star N) - H^0(f^\star N))$, where f is the universal twisted stable map. Since N is trivial (Lemma 6.1), we have $H^1(f^\star N) = 0$ and $e_\mathbb{T}(H^0)$ is identified with $e_\mathbb{T}(N)$. The result follows. □

Corollary 6.5. *The pullback $\iota^\star$ gives an isomorphism*

$$QH^*_{\mathbb{T},big}(X_\theta) \cong QH^*_{\mathbb{T},big}(X_\mathcal{A}).$$

Proof. For $\gamma_1, \gamma_2 \in H^*_{\mathrm{CR},\mathbb{T}}(X_\theta)$, it suffices to show that

$$\iota^\star(\gamma_1 \star_t \gamma_2) = \iota^\star\gamma_1 \star_{\iota^\star t} \iota^\star\gamma_2 \tag{6.1}$$

for $t \in H^*_{\mathrm{CR},\mathbb{T}}(X_\theta)$. Given $d \in H_2(X_\theta)$, the contribution from d to the LHS of (6.1) is the sum over n of

$$\left\langle \gamma_1, \underbrace{t, \cdots, t}_{n}, \gamma_2, \gamma \right\rangle_{0,n+3,d}^{X_\theta} \mathrm{PD}_{X_\theta}(\gamma),$$

where $\mathrm{PD}_{X_\theta}(\gamma)$ is the orbifold Poincare dual of γ. By Proposition 6.4, this is equal to

$$\left\langle \iota^\star\gamma_1, \underbrace{\iota^\star t, \cdots, \iota^\star t}_{n}, \iota^\star\gamma_2, \iota^\star\gamma \right\rangle_{0,n+3,d}^{X_\mathcal{A}} \frac{\mathrm{PD}_{X_\theta}(\gamma)}{e_\mathbb{T}(N)}.$$

By localization, we have the following relation for orbifold Poincaré pairings:

$$\int_{IX_\theta} a \cup b = \int_{IX_\theta^\mathbb{T}} \frac{a \cup b}{e_\mathbb{T}(N_{IX_\theta^\mathbb{T}/IX_\theta})} = \int_{IX_\mathcal{A}^\mathbb{T}} \frac{\iota^\star a \cup \iota^\star b}{e_\mathbb{T}(N_{IX_\mathcal{A}^\mathbb{T}/IX_\mathcal{A}}) \cdot e_\mathbb{T}(N)} = \int_{IX_\mathcal{A}} \frac{\iota^\star a \cup \iota^\star b}{e_\mathbb{T}(N)},$$

where we use [17, Proposition 3.9], which identifies $N_{IX_\mathcal{A}/IX_\theta}$ with the pullback of N to $IX_\mathcal{A}$ under $IX_\mathcal{A} \to X_\mathcal{A}$. Therefore $\mathrm{PD}_{X_\mathcal{A}}(\iota^\star\gamma) = \iota^\star \frac{\mathrm{PD}_{X_\theta}(\gamma)}{e_T(N)}$ and we get

$$\begin{aligned}\left\langle \gamma_1, \underbrace{t, \cdots, t}_{n}, \gamma_2, \gamma \right\rangle_{0,n+3,d}^{X_\theta} \iota^\star\mathrm{PD}_{X_\theta}(\gamma) &= \left\langle \iota^\star\gamma_1, \underbrace{\iota^\star t, \cdots, \iota^\star t}_{n}, \iota^\star\gamma_2, \iota^\star\gamma \right\rangle_{0,n+3,d}^{X_\mathcal{A}} \iota^\star\frac{\mathrm{PD}_{X_\theta}(\gamma)}{e_T(N)} \\ &= \left\langle \iota^\star\gamma_1, \underbrace{\iota^\star t, \cdots, \iota^\star t}_{n}, \iota^\star\gamma_2, \iota^\star\gamma \right\rangle_{0,n+3,d}^{X_\mathcal{A}} \mathrm{PD}_{X_\mathcal{A}}(\iota^\star\gamma)\end{aligned}$$

which is the degree d contribution to the RHS of (6.1). □

Remark 6.6 (Higher genus Gromov-Witten invariants)**.** We briefly discuss what happens in higher genus. Results in toric Gromov-Witten theory [14] shows that $QH_{\mathbb{T},\mathrm{big}}(X_\theta)$ is semi-simple. By Corollary 6.5, the same is true for $QH_{\mathbb{T},\mathrm{big}}(X_{\mathcal{A}})$. The Givental-Teleman reconstruction [20], [45] applies to determine higher genus Gromov-Witten invariants of X_θ and $X_{\mathcal{A}}$, provided the R-calibrations can be specified.

Since equivariant Gromov-Witten theory is not conformal, we follow the toric case to specify R by computing its degree 0 limit. The argument for Proposition 6.4 applies to show that the degree 0, genus g Gromov-Witten invariants of X_θ are equal to degree 0, genus g Gromov-Witten invariants of $X_{\mathcal{A}}$ twisted by $(N, e_{\mathbb{T}}(-)^{-1})$, which amounts to include Hodge classes. Since N is trivial, we may apply results of [18] to remove the Hodge classes. Writing in Givental's formalism [20], this leads to $R^{X_\theta}_{\deg=0} = \Delta R^{X_{\mathcal{A}}}_{\deg=0}$, where Δ is an operator that can be explicitly written down. Together with ancestor/descendant relation, we we should obtain an equality of descendant potentials: $D_{X_\theta} = \hat{\Delta} D_{X_{\mathcal{A}}}$. We leave the details to the interested readers.

6.2. **Proof of Theorem 1.2.** By Corollary 6.5, the small quantum ring $QH^*_{\mathbb{T}}(X_{\mathcal{A}})$ is isomorphic to the small quantum ring $QH^*_{\mathbb{T}}(X_\theta)$ of the associated Lawrence toric Deligne-Mumford stack X_θ. Theorem 1.2 is obtained by deriving a presentation of $QH^*_{\mathbb{T}}(X_\theta)$ using known results in toric Gromov-Witten theory.

Recall the construction of Lawrence toric Deligne-Mumford stack X_θ in §2.1.2, the toric fan Σ_θ is constructed from the Gale dual of the map

$$\mathbb{Z}^m \oplus \mathbb{Z}^m \stackrel{(\beta^\vee, -\beta^\vee)}{\longrightarrow} DG(\beta).$$

Let $D_1, \cdots, D_m, D'_1, \cdots, D'_m$ be the images of the above map, which are the toric divisors of X_θ. Then the Chern class of the tangent bundle if given by $\sum_{i=1}^m (D_i + D'_i) = 0$, hence the toric Deligne-Mumford stack X_θ is Calabi-Yau. As explained in [26], it follows that the toric mirror theorem of [14] implies that the $\mathbb{T}$-equivariant quantum cohomology ring of X_θ with quantum parameters in $H^{\leq 2}_{orb}$ is isomorphic to Batyrev's quantum ring, taking into account of the *mirror map* along $H^{\leq 2}_{orb}$. A direct computation shows that when the quantum parameters are restricted to $H^{\leq 2} \subset H^{\leq 2}_{orb}$, Batyrev's quantum ring is isomorphic to the ring in Theorem 1.2. Therefore it remains to show that the mirror map is the identity when restricted to $H^{\leq 2}$.

Recall that the Gale dual map is

$$\beta_L : \mathbb{Z}^{2m} \to N_L,$$

where $\overline{N}_L$ is a lattice of dimension $2m - (m-d)$. The map β_L is given by the integral vectors $\{b_{L,1}, \cdots, b_{L,m}, b'_{L,1}, \cdots, b'_{L,m}\}$ and β_L is called the Lawrence lifting of β. From Remark 2.3 in [28], $\{b_{L,1}, \cdots, b_{L,m}, b'_{L,1}, \cdots, b'_{L,m}\}$ are the vectors

$$\{(b_1, e_1), \cdots, (b_m, e_m), (0, e_1), \cdots, (0, e_m)\},$$

where $\{e_i\}$ are the standard bases of $\mathbb{Z}^m$. The toric divisors D_i, D'_i define equivariant cohomology classes $u_i, u'_i \in H^2(X_\theta)$, and we have $u'_i = \hbar - u_i$.

We only need to prove that in this case the mirror map is trivial.

The mirror map restricted to $H^{\leq 2}$ is calculated in [15, Section 6.3]. To study the mirror map, we need to understand elements of the set $\mathbb{K}_{\mathrm{eff}}$, which is the nef cone. By

definition an element in $\mathbb{K}_{\mathrm{eff}}$ is of the form

$$\beta = \sum_{i=1}^{m} c_i e_i + \sum_{i=1}^{m} c_i' e_i' \in \mathbb{Z}^m \oplus \mathbb{Z}^m$$

such that

$$\sum_i c_i b_{L,i} + \sum_i c_i' b_{L,i}' = 0,$$

i.e.

$$\sum_i c_i b_i = 0, \text{ and } c_i + c_i' = 0 \text{ for any } i.$$

Moreover, $c_i = \langle D_i, \beta \rangle$, $c_i' = \langle D_i', \beta \rangle$. If $\beta \in \Omega_i^{X_\theta}$, where $\Omega_i^{X_\theta}$ is defined on page 39 of [15], then we have $\langle D_i, \beta \rangle < 0$, $\langle D_j, \beta \rangle \geq 0$ for $D_j \neq D_i$. For $j \neq i$, we have $\langle D_j, \beta \rangle + \langle D_j', \beta \rangle = 0$. Since both are nongenative, we have $\langle D_j, \beta \rangle = \langle D_j', \beta \rangle = 0$ for $j \neq i$. On the other hand, the definition of β implies that

$$\sum_k \langle D_k, \beta \rangle b_k = 0 \in N.$$

Because $\langle D_j, \beta \rangle = 0$ for $j \neq i$, this reduces to $\langle D_i, \beta \rangle b_i = 0$, which contradicts the requirement that $\langle D_i, \beta \rangle < 0$. So $\Omega_i^{X_\theta} = \emptyset$. Similarly the set corresponding to D_i' is empty. We thus conclude that the mirror map restricted to $H^{\leq 2}$ is trivial. This completes the proof of Theorem 1.2.

We end with calculating the small equivariant quantum cohomology rings for two examples. In the second example we also explain how to get the quantum Stanley-Reisner relation from the quantum product formula in Theorem 1.1.

Example 6.7. Let $\mathcal{A} = (N, \beta, \theta)$ be the stacky hyperplane arrangement given by $N = \mathbb{Z}$, $\beta : \mathbb{Z}^2 \to N$ is given by $\{-1, 1\}$, and $\theta = 1$ in $DG(\beta) = \mathbb{Z}$. The hypertoric Deligne-Mumford stack is $T^*\mathbb{P}^1$, which is a closed substack of the Lawrence toric Deligne-Mumford stack $X_\theta = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let u_1, u_2 be the hyperplane classes of $\mathbb{P}^1$, and $\hbar$ the first Chern class of the extra $\mathbb{C}^\times$-action. Our formula (1.1) is:

$$u_1 \star u_2 = \frac{Q^\ell}{1 - Q^\ell} (\hbar - u_1) \cdot (\hbar - u_2).$$

Then using this formula we can find that the quantum ring

$$QH^*_{\mathbb{T}}(X_{\mathcal{A}}) = \frac{\mathbb{Q}[u_1, u_2, \hbar]}{(u_1 \star u_2 - Q^\ell (\hbar - u_1) \star (\hbar - u_2))}.$$

This was computed in [40].

Example 6.8. Let $\mathcal{A} = (N, \beta, \theta)$ be the stacky hyperplane arrangement given by $N = \mathbb{Z}$, $\beta : \mathbb{Z}^2 \to N$ is given by $\{-2, 1\}$, and $\theta = 1$ in $DG(\beta) = \mathbb{Z}$. The hypertoric Deligne-Mumford stack in this case is $X_{\mathcal{A}} = T^*\mathbb{P}(1,2)$, which is a closed substack of the Lawrence toric Deligne-Mumford stack $X_\theta = \mathcal{O}_{\mathbb{P}(1,2)}(-1) \oplus \mathcal{O}_{\mathbb{P}(1,2)}(-2)$.

Let u_1, u_2 be the hyperplane classes of $\mathbb{P}(1,2)$, and $\hbar$ the first Chern class of the extra $\mathbb{C}^\times$-action in the $\mathbb{T} = (\mathbb{C}^\times)^2 \times \mathbb{C}^\times$-action on $X_{\mathcal{A}}$. The quantum Batyrev ring is

$$QH^*_{\mathbb{T}}(X_{\mathcal{A}}) = \frac{\mathbb{Q}[u_1, u_2, \hbar]}{(u_1 \star (u_2)^2 - Q^{2\ell} (\hbar - u_1) \star (\hbar - u_2)^2)},$$

which is the ring in Theorem 1.2.

Let $\mathbb{1}_{1/2}$ be the identity class of the twisted sector $(X_{\mathcal{A}})_{1/2} = B\mu_2$. We explain that our formula in Theorem 1.1 generate this class. From Theorem 1.1, the quantum product by divisor is:

$$u_1 \star u_2 = \frac{Q^{2\ell}}{1-Q^{2\ell}}(\hbar - u_1)\cdot(\hbar - u_2) + \frac{-Q^{\ell}}{1-Q^{2\ell}}\mathbb{1}_{1/2}\cdot \hbar.$$

Hence we get the class $\mathbb{1}_{1/2}$ up to $\hbar$. Now we explain that from formulas of quantum product by divisors we can get the relation in the quantum ring. Using u_2 to do quantum product on the above formula, we get

$$u_1 \star u_2 \star u_2 = \frac{-Q^{\ell}}{1-Q^{2\ell}}\mathbb{1}_{1/2}\cdot \hbar \star u_2,$$

since by Theorem 1.1, when doing quantum product with $(\hbar - u_1)\cdot(\hbar - u_2)$, the integration of $(\hbar - u_1)\cdot(\hbar - u_2)$ over $\mathbb{P}(1,2)$ is zero. Hence we get:

$$u_1 \star u_2 \star u_2 = \frac{-Q^{\ell}}{1-Q^{2\ell}}\cdot\frac{-Q^{3\ell}}{1-Q^{2\ell}}(\hbar - u_1)\cdot(\hbar - u_2)^2.$$

Similarly we have:

$$(\hbar - u_1)\star(\hbar - u_2) = (\hbar - u_1)\cdot(\hbar - u_2) + \frac{Q^{2\ell}}{1-Q^{2\ell}}(\hbar - u_1)\cdot(\hbar - u_2) + \frac{-Q^{\ell}}{1-Q^{2\ell}}\mathbb{1}_{1/2}\cdot\hbar.$$

Hence

$$(\hbar - u_1)\star(\hbar - u_2)\star(\hbar - u_2) = \frac{-Q^{\ell}}{1-Q^{2\ell}}\cdot\frac{-Q^{\ell}}{1-Q^{2\ell}}(\hbar - u_1)\cdot(\hbar - u_2)^2.$$

Then the relation is obtained by multiplying $Q^{2\ell}$ on above.

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